International Journal of Theoretical Physics, Vol. 33, No. 5, 1994

Physical Models of Cognition

Michail Zak[1]

Received August 13, 1993

This paper presents and discusses physical models for simulating some aspects of neural intelligence, and, in particular, the process of cognition. The main departure from the classical approach here is in utilization of a terminal version of classical dynamics introduced by the author earlier. Based upon violations of the Lipschitz condition at equilibrium points, terminal dynamics attains two new fundamental properties: it is spontaneous and nondeterministic. Special attention is focused on terminal neurodynamics as a particular architecture of terminal dynamics which is suitable for modeling of information flows. Terminal neurodynamics possesses a well-organized probabilistic structure which can be analytically predicted, prescribed, and controlled, and therefore which presents a powerful tool for modeling real-life uncertainties. Two basic phenomena associated with random behavior of neurodynamic solutions are exploited. The first one is a stochastic attractor—a stable stationary stochastic process to which random solutions of a closed system converge. As a model of the cognition process, a stochastic attractor can be viewed as a universal tool for generalization and formation of classes of patterns. The concept of stochastic attractor is applied to model a collective brain paradigm explaining coordination between simple units of intelligence which perform a collective task without direct exchange of information. The second fundamental phenomenon discussed is terminal chaos which occurs in open systems. Applications of terminal chaos to information fusion as well as to explanation and modeling of coordination among neurons in biological systems are discussed. It should be emphasized that all the models of terminal neurodynamics are implementable in analog devices, which means that all the cognition processes discussed in the paper are reducible to the laws of Newtonian mechanics.

1. INTRODUCTION

The process of human cognition represented by information flows on a certain time scale can be viewed as a dynamical process. On a time scale of seconds and minutes, this process has a sequential character (Rumelhart, 1987) (acceptance, rejection, or replacement of ideas). On a much smaller

[1]Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91109.

0020-7748/94/0500-1113$07.00/0

time scale (recognition of words and images) it can be represented by parallel distributed processing. Hence, an underlying dynamical system which models the cognition processes should have a complex multiscale structure. But can such a dynamical system be derived from physical laws? The answer to this fundamental question divides the scientific community into reductionists and their opponents. According to the reductionists, there is an intrinsic unity of science, and there are no independent levels of modeling, i.e., all natural phenomena are reducible to physical laws. Most biologists and cognitive scientists express doubts about such a possibility. In this paper, remaining neutral to philosophical aspects of the problem, we demonstrate the possibility to develop such a dynamical system which, being implementable in analog devices, can simulate some aspects of the processes of human cognition.

2. BACKGROUND

2.1. General Remarks

One of the oldest and most challenging problems in cognitive science is to understand the relationships between cognitive phenomena and brain functioning. Since ancient times the mystery of mind has attracted philosophers, neuroscientists, psychologists, and later, mathematicians, physicists, etc. In line with attempts to understand and to simulate brain activity itself, there have been many successes in developments of brain-style information processing devices which focus on brain-inspired modeling rather than modeling of the brain as a part of the human body. The most powerful information processing device of this kind is the digital computer, which has revolutionized science and technology in our century and even changed the lifestyle of the whole society. The digital computer became the first candidate for human brain modeling. Artificial intelligence researchers predicted that "thinking machines" will take over our mental work. Futurologists have proclaimed the birth of a new species, machina sapiens, that will share our place as the intelligent sovereigns of our earthly domain.

Notwithstanding some achievements in "thinking machines" developments, it seems very unlikely that digital computers, with their foundations on rigid, cold logic and full predictability, can model even the simplest biological systems, which are flexible, creative, and, to a certain degree, irrational. In addition, the main brain characteristics which contribute to information processing are different from those of digital computers. Indeed, neurons, as the basic hardware of the brain, are a million times slower than the information processing elements of serial computers. This slow speed is compensated by an extremely large number (up to a hundred of billions) of neurons as active processing units which are highly interconnected. Hence, the brain succeeds through massive parallelism of a large number of slow neurons, and therefore the mechanisms of mind are most likely best understood as resulting from the cooperative activity of very many relatively simple processing units working in parallel rather than by fast, but sequential processing units of digital computers. There is much indirect evidence that the structures of the computational procedures in digital computers and brains are also different: instead of calculating a solution using sequences of rigid rules, the primary mode of computation in the brain is rather associated with a relaxation procedure, i.e., with settling into a solution in the same way in which a dynamical system converges to an attractor. Another difference between a digital computer and the brain is in the mechanisms of learning and memory storing. A number of facts suggest that the knowledge is in the connections between the neurons, rather than in the neurons themselves, while these connections have a clear geometric and topological structure. Such a distributed memory storage is responsible for the graceful degradation phenomenon when the system performance gradually deteriorates as more and more neural units are destroyed, but there is no single critical point where performance breaks down. Based upon this kind of representation of the distributed memory, the learning procedure can be understood as gradual modifications of the connection strengths during a relaxation-type dynamical process.

Along with the abstract model of a computer as a formal machine that could be programmed to carry out an effective procedure, introduced by Alan Turing in 1936, another potential candidate for brain simulation has been developed: McCulloch and Pitts (1943) offered a formal model of the neuron as a threshold logic unit. They demonstrated that each Turing machine program could be implemented using a finite network of their formal neurons. In support to the idea of neural networks, Hebb's (1949) work provided the inspiration for many computation models of learning. However, it took about 30 years until the neural networks became a potential competitor to digital computers in regard to simulation of brain performance.

Two main factors significantly contributed to the "second birth" of the neural networks. The first factor is associated with the pioneering work of Carver Mead on the design of neural networks and their implementation in analog VLSI systems. In his work he has shown how the powerful organizing principles of nervous systems can be realized in silicon integrated circuits. The second factor is based upon the progress in dynamical system theory. In the past, most theoretical studies of dynamical systems have been concerned with modeling of energy transformations. However, in recent years several attempts were made to exploit the phenomenology of nonlinear dynamical systems for information processing as an alternative to the traditional paradigm of finite-state machines.

There is much evidence coming from the analysis of electroencephalogram data that human brain activity resembles a dissipative nonlinear adaptive system. In contradistinction to finite-state machines which operate by simple bits of information, the nonlinear dynamics paradigm operates in terms of complex "blocks" of information which resemble patterns of practical interest.

2.2. Neural Net As a Dynamical System

The current artificial neural nets can be considered as massively parallel adaptive dynamical systems modeled on the general features of biological neural networks that are intended to interact with the objects of the real world in the same way the biological systems do.

As a dynamical system, a neural net is characterized by nonlinearity and dissipativity, which provide the existence of at least several attractors. There are many different modifications of neural nets. In this paper we will be interested only in those neural net architectures which do not contain any man-made devices (such as digital devices) and therefore are suitable for circuit implementations. Such neural nets (which in the literature are called continuously updated recurrent neural nets) can be represented by the following dynamical system:

$$\tau_i \dot{x}_i = -x_i + \sigma\left(\sum_j T_{ij} x_j\right), \qquad \tau_i > 0 \tag{1}$$

where x_i are state variables, or mean soma potentials, characterizing the neuron activities, T_{ij} are constant control parameters representing the weights of synaptic interconnections, τ_i are suitable time constants, and $\sigma(\cdot)$ is a sigmoid function having a saturated nonlinearity [usually $\sigma(x) = \tanh \beta x$, where $\beta = \text{const} > 0$ is an additional control parameter].

An invariant characterizing the local dissipativity of the system (1) is expressed explicitly via its parameters:

$$\operatorname{div} \dot{x} = \sum_i \frac{1}{\tau_i}\left(-1 + \frac{\beta T_{ii}}{\cosh^2 \sum_j T_{ij} x_j}\right) \tag{2}$$

A necessary (but not sufficient) condition that the system (1) has attractors is that there are some domains in phase space where the invariant (2) is negative.

If the matrix T_{ij} is symmetric

$$T_{ij} = T_{ji} \tag{3}$$

then equations (1) can be represented in the form of a gradient system, and therefore it can have only static attractors. In the basin of a static attractor, the invariant (2) must be negative.

Since the system (1) is nonlinear, it can have more than one attractor; consequently, in some domains of phase space, the invariant (2) may be positive or zero.

Equation (1) presents the neural net in its "natural" form in the sense that x_i and T_{ij} correspond to physical parameters: neuron potentials and synaptic interconnections, respectively. However, it is important to emphasize that the relationship between the invariants of the "vector" u_i and the "tensor" T_{ij} are not preserved by the coordinate transformation, i.e., equation (1) does not possess an invariant tensor structure. Consequently, the column u_i and the matrix T_{ij} cannot be treated as a vector and tensor, respectively.

In most applications, the neural nets performance is associated with convergence to attractors (pattern recognition, optimization, decision making, control, associative memory, generalization, etc.). The locations of attractors and their basins in phase space can be prescribed by an appropriate choice of the synaptic weights T_{ij} i.e., by solving inverse dynamical problems. However, since the dimensionality of neural nets is usually very high (in biological systems it is of order of 10^{11} with the number of synaptic interconnections of order of 10^{15}), the straightforward analytical approach can be very expensive and time consuming. An alternative way to select synaptic weights in order to do specific tasks was borrowed from biological systems. It is based upon iterative adjustments of T_{ij} as a result of comparison of the net output with known correct answers (supervised learning) or as a result of creating of new categories from the correlations of the input data when correct answers are not known (unsupervised learning). Actually the procedure of learning is implemented by another dynamical system with the state variables T_{ij} which converges to certain attractors representing the desired synaptic weights $\dot{T}_{ij}$.

2.3. Limitations of Classical Approach

The biggest promise of artificial neural networks as computational tools lies in the hope that they will resemble the information processing in biological systems. Notwithstanding many successes in this direction, it is rapidly becoming evident that current models are characterized by a number of limitations. We will analyze these limitations using the additive model as a typical representative of artificial neural networks:

$$\dot{x}_i + x_i = \sum_{j=1}^{n} T_{ij}\sigma(x_j) + I_i, \qquad i = 1, 2, \ldots, n \tag{4}$$

in which $x_i(t)$ is the mean soma potential of the ith neuron, T_{ij} are constant synaptic interconnections, $\sigma(x)$ is a sigmoid function, and I_i is an external input.

First, the neuron performance in this model is collective, but not parallel: any small change in the activity of an ith neuron instantaneously affects all other neurons:

$$\frac{\partial \dot{x}_i}{\partial x_i} = \frac{d\sigma}{dx_j} T_{ij} \neq 0 \tag{5}$$

In contrast, all biological systems exhibit both collective and parallel performances. For instance, the right and the left hands are mechanically independent (i.e., their performance is parallel), but at the same time, their activity is coordinated by the brain; that makes their performance collective.

Second, the performance of the model (4) is fully prescribed by initial conditions. The system never "forgets" these conditions: it carries their "burden" up to $t \to \infty$. In order to change the system performance, the external input must overpower the "inertia of the past." In contrast the biological systems are much more flexible: they can forget (if necessary) the past, adapting their behavior to environmental changes.

Third, the features characterizing the system (4) are of the same scale: they are insulated from the microworld by a large range of scales. At the same time, biological systems involve mechanisms that span the entire range from the molecular to the macroscopic.

Can these limitations be removed within the framework of classical dynamics? The answer is no. Indeed, all the systems considered here are based on classical dynamics and satisfy the Lipschitz condition which guarantees the uniqueness of the solutions subject to prescribed sets of initial conditions. For the system (1) this condition requires that all the derivatives $\partial \dot{x}_i / \partial x_j$ exist and are bounded:

$$\left| \frac{\partial \dot{x}_i}{\partial x_j} \right| < \infty \tag{6}$$

The uniqueness of the solution

$$x_i = x_i(t, x_1, \ldots, x_n), \qquad i = 1, 2, \ldots, n \tag{7}$$

subject to the initial conditions $x_i(0, x_1, \ldots, x_n) = \mathring{x}_i$ can be considered as a mathematical interpretation of the rigid, predictable behavior of the corresponding dynamical system.

Actually, all the limitations of the current neural net models mentioned above are inevitable consequences of the Lipschitz condition (6) and therefore of determinism of classical dynamics.

2.4. Terminal Dynamics

It has long been recognized that classical deterministic dynamics is not suitable for capturing the truly dynamical behavior of real-world applications, and in particular the dynamics of information flows, the dynamics of multichoice human behavior, etc. A primary reason is that this kind of dynamical process is characterized by uncertainties which arise in a dynamic setting (for instance, when information that will be needed in subsequent decision stages is not yet available to the decision-maker). Uncertainties are described for purposes of mathematical analysis by probability theory, and the dynamics simulating the evolution of uncertainties acquires stochastic properties.

Turning to governing equations of classical dynamics,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} - \frac{\partial R}{\partial \dot{q}_i}, \qquad i = 1, 2, \ldots, n \tag{8}$$

where L is the Lagrangian, q_i, $\dot{q}_i$ are the generalized coordinates and velocities, and R is the dissipation function, one should recall that the structure of $R(\dot{q}_i, \ldots, \dot{q}_n)$ is not prescribed by Newton's laws: some additional assumptions are to be made in order to define it. The "natural" assumption (which has never been challenged) is that these functions can be expanded in Taylor series with respect to equilibrium states: $\dot{q}_i = 0$. Obviously this requires the existence of the derivative:

$$\left|\frac{\partial^2 R}{\partial \dot{q}_i\, \partial \dot{q}_i}\right| < \infty \qquad \text{at} \quad \dot{q}_i \to 0$$

A departure from that condition was proposed in Zak (1992), where the following dissipation function was introduced:

$$R = \frac{1}{k+1} \sum_i \alpha_i \left| \sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j \right|^{k+1} \tag{9}$$

in which

$$k = \frac{p}{p+1} < 1, \qquad p \gg 1 \tag{10}$$

where p is a large, odd number.

By selecting large p, one can make k close to 1, so that equation (9) is almost identical to the classical one (when $k = 1$) everywhere excluding a small neighborhood of the equilibrium point $\dot{q}_j = 0$, while at this point

$$\left|\frac{\partial^2 R}{\partial \dot{q}_i\, \partial \dot{q}_j}\right| \to \infty \qquad \text{at} \quad \dot{q}_j \to 0 \tag{11}$$

Hence, the Lipschitz condition is violated; the friction force $F_i = -\partial R/\partial \dot{q}_i$ grows sharply at the equilibrium point, and then it gradually approaches its "classical" value. This effect can be interpreted as a mathematical representation of a jump from static to kinetic friction, when the dissipation force does not vanish with the velocity.

It appears that this "small" difference between the friction forces at $k = 1$ and $k < 1$ leads to fundamental changes in Newtonian dynamics. In order to demonstrate it, we will consider the relationship between the total energy E and the dissipation function R:

$$\frac{dE}{dt} = -\sum_i \dot{q}_i \frac{\partial R}{\partial \dot{q}_i} = -(k+1)R \tag{12}$$

Within a small neighborhood of an equilibrium state (where the potential energy can be set zero) the energy E and the dissipation function R have the order, respectively,

$$E \sim \dot{q}_i^2, \qquad R \sim \dot{q}_i^{k+1} \qquad \text{at} \quad E \to 0$$

Hence, the asymptotic form of equation (5) can be presented as

$$\frac{dE}{dt} = AE^{(k+1)/2} \qquad \text{at} \quad E \to 0, \qquad A = \text{const} \tag{13}$$

If $A > 0$ and $k < 1$, the equilibrium state $E = 0$ is an attractor where the Lipschitz condition ($|d\dot{E}/dE| \to \infty$ at $E \to 0$) is violated. Such a terminal attractor is approached by the solution originating at $E = \Delta E_0 > 0$, in finite time:

$$t_0 = \int_{\Delta E_0}^{0} \frac{dE}{AE^{(k+1)/2}} = \frac{2\Delta E_0^{(1-k)/2}}{(1-k)|A|} < \infty \tag{14}$$

Obviously, this integral diverges in classical case $k \geq 1$, where $t_0 \to \infty$. The motion described by equation (13) has a singular solution $E \equiv 0$ and a regular solution

$$E = \left[\Delta E_0^{(1-k)/2} + \frac{1}{2} A(1-k)t\right]^{2/(1-k)} \tag{15}$$

In a finite time this motion can reach the equilibrium and switch to the singular solution $E \equiv 0$, and this switch is irreversible (Fig. 1).

The coefficient k can be found from experimental observations of the time t_0.

As is well known from the dynamics of nonconservative systems, dissipative forces can destabilize the motion when they feed external energy into the system (the transmission of energy from laminar to turbulent flow in fluid dynamics, or from rotations to oscillations in dynamics of flexible

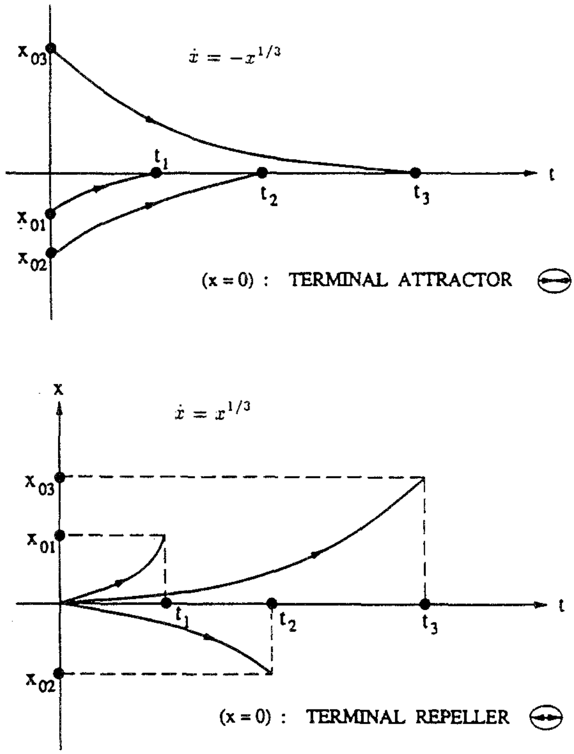

Fig. 1. Behavior of terminal attractor and repeller.

systems). In terms of equation (13) this would mean that $A > 0$, and the equilibrium state $E = 0$ becomes a terminal repeller (Zak, 1992).

If the initial condition is infinitely close to this repeller, the transient solution will escape it during a finite time period:

$$t_0 = \int_{\varepsilon \to 0}^{\Delta E_0} \frac{dE}{AE^{(k+1)/2}} = \frac{2\Delta E_0^{(1-k)/2}}{(1-k)A} < \infty$$

while for a regular repeller, the time would be infinite (Fig. 1).

Expressing equation (13) in terms of velocity at $i = 1$, $\dot{q}_1 = v$,

$$\dot{v} = Bv^k, \qquad B = \text{const} > 0$$

one arrives at the following solution:

$$v = \pm\{[B(1-k)t]^{p+2}\}^{1/2} \tag{16}$$

As in the case of a terminal attractor, here the motion is also irreversible: the time-backward motion obtained by formal time inversion

$t \to -t$ in equation (16) is imaginary, since p is an odd number [see equation (10)].

In addition, terminal repellers possess even more surprising characteristics: the solution (16) becomes totally unpredictable. Indeed, two different motions described by the solution (11) are possible for "almost the same" ($v_0 = +\varepsilon \to 0$, or $v_0 = -\varepsilon \to 0$ at $t \to 0$) initial conditions.

Thus, a terminal repeller represents a vanishingly short, but infinitely powerful "pulse of unpredictability" which is pumped into the system via terminal dissipative forces. Obviously failure of the uniqueness of the solution here results from the violation of the Lipschitz condition at $v = 0$.

Terminal dynamics can be introduced as a set of nonlinear ordinary differential equations of the form

$$\dot{x}_i = v_i^k(x_1, x_2, \ldots, x_n), \qquad i = 1, 2, \ldots, n \tag{17}$$

in which

$$\left|\frac{\partial v_i}{\partial x_j}\right| < \infty \tag{18}$$

and $k < 1$.

Therefore,

$$\left|\frac{\partial \dot{x}_i}{\partial x_j}\right| = kv^{(k-1)}(x_1, \ldots, x_n)\left|\frac{\partial v_i}{\partial x_i}\right| \to \infty \qquad \text{if} \quad \dot{x}_i \to 0 \tag{19}$$

and the Lipschitz condition is violated at all the equilibrium points

$$\dot{x}_i = 0$$

As in the classical case, the equilibrium points are attractors if the real parts of the eigenvalues of the matrix

$$m = \left\|\frac{\partial v_i}{\partial x_j}\right\| \tag{20}$$

are negative, that is,

$$\operatorname{Re} \lambda_i < 0 \tag{21}$$

and are repellers if some of the eigenvalues have positive real parts.

Basic mathematical and physical aspects of terminal dynamics are discussed in (Zak (1988, 1989*a–c*, 1990*a,b*, 1991*a,b*, 1992, 1993*a*). Recently an analog VLSI circuit was designed which, when operated in the subthreshold domain, models the terminal attractor/repeller phenomena (Cetin *et al.*, 1991).

3. TERMINAL NEURODYNAMICS

3.1. General Remarks

Terminal neurodynamics is a particular version of terminal dynamics which is suitable for modeling information flows. For that purpose, it has a well-organized probabilistic structure which can be prescribed, predicted, and controlled. Terminal neurodynamics and its applications to information processing are discussed in Zak (1990*b*, 1991*a*,*b*, 1992, 1993*a*).

We should emphasize the fundamental difference between the probabilistic properties of terminal dynamics and those of stochastic or chaotic differential equations. Indeed, the randomness of stochastic differential equations is caused by random initial conditions, random forces, or random coefficients; in chaotic equations small (but finite!) random changes of initial conditions are amplified by the mechanism of instability. But in both cases the differential operator itself remains deterministic. In contradistinction to that, in terminal dynamics randomness results from the violation of the uniqueness of the solution at equilibrium points, and therefore the differential operator itself generates random solutions.

Terminal neurodynamics is based upon a physical implementation of the random walk paradigm.

3.2. Random Walk Paradigm

A random walk is a stochastic process where changes occur only at fixed times; it represents the position at time t_m of a particle taking a random "step" x_m independently of its previous ones.

Let us start with the following dynamical system:

$$\dot{x} = \gamma \sin^{1/3} \frac{\sqrt{\omega}}{\alpha} x \sin \omega t, \qquad \gamma = \text{const}, \qquad \omega = \text{const}, \qquad \alpha = \text{const} \tag{22}$$

It can be verified that at the equilibrium points

$$x_m = \frac{\pi m \alpha}{\sqrt{\omega}}, \qquad m = \ldots, -2, -1, 0, 1, 2, \ldots \tag{23}$$

the Lipschitz condition is violated:

$$\partial \dot{x}/\partial x \to \infty \qquad \text{at} \quad x \to x_m \tag{24}$$

If $x = 0$ at $t = 0$, then during the first period

$$0 < t < \pi/\omega \tag{25}$$

the point $x_0 = 0$ is a terminal repeller since $\sin \omega t > 0$ and the solution at this point splits into two (positive and negative) branches whose divergence is unbounded (Zak, 1992) [equation (16)]. Consequently, with an equal probability, x can move in the positive or the negative direction. For the sake of concreteness, we will assume that it moves in the positive direction. Then the solution will approach the second equilibrium point $x_1 = \pi\alpha/\sqrt{\omega}$ at

$$t^* = \frac{1}{\omega} \arccos\left[1 - \frac{B(1/3, 1/3)}{2^{1/3}} \frac{\alpha\sqrt{\omega}}{\gamma}\right] \tag{26}$$

in which B is the beta function.

It can be verified that the point x_1 will be a terminal attractor at $t = t_1$ if

$$t_1 \leq \frac{\pi}{\omega}, \qquad \text{i.e.,} \quad \text{if } \frac{\gamma}{\alpha} \geq \frac{B(1/3, 1/3)}{2^{4/3}} \sqrt{\omega} \tag{27}$$

Therefore, x will remain at the point x_1 until it becomes a terminal repeller, i.e., until $t > t_1$. Then the solution splits again: one of two possible branches approaches the next equilibrium point $x_2 = 2\pi\alpha/\sqrt{\omega}$, while the other returns to the point $x_0 = 0$, etc. The periods of transition from one equilibrium point to another are all the same and are given by equation (26) (Fig. 2).

It is important to notice that these periods t^* are bounded only because of the failure of the Lipschitz condition at the equilibrium points. Otherwise they would be unbounded, since the time of approaching a

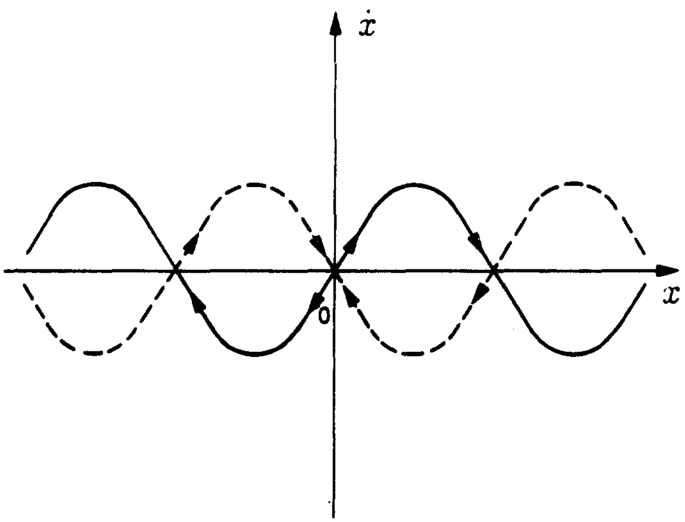

Fig. 2. Oscillations about the attractor $X = 0$.

regular attractor (as well as the time of escaping a regular repeller) is infinite.

Thus, the evolution of x prescribed by equation (22) is totally unpredictable: it has 2^m different scenarios where $m = E(t/t^*)$ (Fig. 3), while any prescribed value of x from equation (23) will appear eventually. (Here E is the integer part of the ratio t/t^*.) This evolution is identical to a random walk, and the probability $f(x, t)$ is governed by the following difference equation:

$$f\left(x, t+\frac{\pi}{\omega}\right)=\frac{1}{2}f\left(x-\frac{\pi\alpha}{\sqrt{\omega}}, t\right)+\frac{1}{2}f\left(x+\frac{\pi\alpha}{\sqrt{\omega}}, t\right) \tag{28}$$

For better physical interpretation we will assume that

$$\frac{\pi\alpha}{\sqrt{\omega}} \ll L, \qquad t^* \ll T, \qquad \text{i.e.,} \quad \omega \to \infty \tag{29}$$

in which L and T are the total length and the total time period of the random walk, respectively. Setting

$$\frac{\pi\alpha}{\sqrt{\omega}} \to 0, \qquad t^* \to 0 \tag{30}$$

one arrives at the Fokker–Planck equation:

$$\frac{\partial f(x, t)}{\partial t}=\frac{1}{2}D^2\frac{\partial^2 f(x, t)}{\partial x^2}, \qquad D^2=\pi\alpha^2 \tag{31}$$

Its unrestricted solution for the initial condition that the random walk starts from the origin $x=0$ at $t=0$

$$f(x, t)=\frac{1}{(2\pi D^2 t)^{1/2}}\exp\left(-\frac{x^2}{2D^2 t}\right) \tag{32}$$

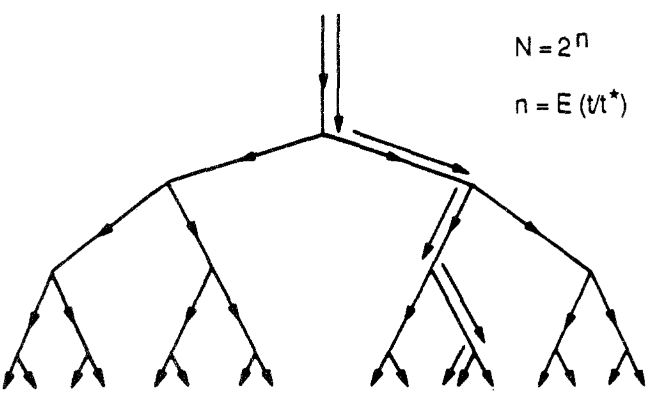

Fig. 3. Unpredictable evolution.

qualitatively describes the evolution of the probability distribution for the dynamical equation (22). It is worth noticing that for the exact solution one should turn to the difference equation (28) since actually $\omega < \infty$.

Equation (28) can be presented in the following operator form:

$$\left[E_t - \frac{1}{2}(E_x + E_x^{-1})\right]f = 0 \tag{33}$$

where E_t and E_x are shift operators:

$$E_t f(x, t) = f(x, t + \tau), \qquad E_x f(x, t) = f(x + h, t), \qquad h = \frac{\pi\alpha}{\sqrt{\omega}} \tag{34}$$

Utilizing the relationships between the shift and the differential operator D

$$E_t^r = e^{r\tau D_t}, \qquad E_x^r = e^{rhD_x}, \qquad D_t = \frac{\partial}{\partial t}, \qquad D_x = \frac{\partial}{\partial x} \tag{35}$$

one can transform from equation (28) to equation (31) if $\omega \to \infty$, i.e., $\tau, h \to 0$.

For further analysis it will be more convenient to modify equation (22) as follows:

$$\dot{x} = \gamma \sin^k\left(\frac{\sqrt{\omega}}{\alpha} x\right) \sin \omega t \tag{36}$$

assuming that

$$k = \frac{1}{2n + 1}, \qquad n \to \infty \tag{37}$$

where n is an integer.

This replacement does not change the qualitative behavior of the dynamical system (37): it changes only its quantitative behavior between the critical points in such a way that one has explicit control over the period of transition from one critical point to another (Zak, 1993*a*).

3.3. Stochastic Attractors in Terminal Dynamics

The dynamical system considered above exhibited an unrestricted random walk. As a result, the probability density of the solutions vanishes at $t \to \infty$. In this section we will describe a new phenomenon—an attraction of the solution to a stationary stochastic process whose density function is uniquely defined by the parameters of the original dynamical system.

We will start with the following one-dimensional dynamical system:

$$\dot{x} = \gamma \sin^k\left(\frac{\sqrt{\omega}}{\alpha} \sin x\right) \sin \omega t \tag{38}$$

It has the following equilibrium points:

$$\overset{*}{x}_m = \arcsin\left(\frac{\pi\alpha}{\sqrt{\omega}} m\right), \qquad m = \ldots, -1, -1, 0, 1, 2, \ldots \tag{39}$$

Obviously now the distances between these points depend upon the number of step m:

$$h_m = \overset{*}{x}_m - \overset{*}{x}_{m-1} = \arcsin\left(\frac{\pi\alpha}{\sqrt{\omega}} m\right) - \arcsin\left[\frac{\pi\alpha}{\sqrt{\omega}}(m-1)\right] \tag{40}$$

Let us introduce a new variable:

$$y = \sin x \tag{41}$$

Then

$$\overset{*}{y}_m = \frac{\pi\alpha}{\sqrt{\omega}} m, \qquad \overset{*}{y}_m - \overset{*}{y}_{m-1} = \frac{\pi\alpha}{\sqrt{\omega}} \tag{42}$$

This means that the probability as a function of y satisfies the following equation:

$$\left[E_t - \frac{1}{2}(E_y + E_y^{-1})\right] f(t, y) = 0 \tag{43}$$

However, in contradistinction to equation (33), here y is bounded:

$$|y| = |\sin x| \leq 1 \tag{44}$$

If the solution of equation (43) subject to the boundary condition (74) is found

$$f = \overset{*}{f}(t, y) \tag{45}$$

then the solution to the original problem is

$$f = \overset{*}{f}[t, y(\sin x)]|\cos x| \tag{46}$$

For better physical interpretation of the solution (46) consider a limit case when

$$\sqrt{\omega} \to \infty, \qquad \text{i.e.,} \quad \tau, h_m \to 0 \tag{47}$$

Then equation (43) transforms to the Fokker–Planck equation:

$$\frac{\partial f}{\partial t} = \frac{1}{2} D^2 \frac{\partial^2 f}{\partial y^2} \tag{48}$$

with the boundary conditions

$$\left.\frac{\partial f}{\partial y}\right|_{y=1} = \left.\frac{\partial f}{\partial y}\right|_{y=-1} = 0 \tag{49}$$

Its solution subject to the initial conditions

$$f(0, y) = \varphi(y), \qquad \varphi(y) \geq 0, \qquad \int_{-1}^{1} \varphi(y)\, dy = 1 \tag{50}$$

is

$$f(t, y) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n e^{-\pi^2 D^2 n^2 t/2} \cos \frac{n\pi}{2}(y+1), \qquad |y| \leq 1 \tag{51}$$

$$a_n = 2 \int_{-1}^{1} \varphi(z) \cos \frac{n\pi}{2}(z+1)\, dz, \qquad n = 1, 2, \ldots \tag{52}$$

and therefore

$$f(t, y) \to \tfrac{1}{2} \quad \text{at} \quad t \to \infty, \quad |y| \leq 1 \tag{53}$$

Returning to the original variable x, one obtains instead of (53)

$$f(x) = 0.5|y'| = 0.5 \cos x, \qquad -\pi/2 < x < \pi/2 \tag{54}$$

Hence, any solution originating within the interval

$$-\pi/2 < x < \pi/2 \tag{55}$$

always approaches the same stationary stochastic process (54), which plays the role of a stochastic attractor.

It should be emphasized that this is a new phenomenon which does not exist in the classical version of nonlinear dynamics. Unlike chaotic attractors, here the probability density can be uniquely controlled by the parameters of the original dynamical system, while the limit stochastic process does not depend upon the initial conditions if they are within the basin of attraction.

Now one can generalize equation (38) by requiring that its solution should have a stochastic attractor with a prescribed density function $f(x)$ under the only restrictions that

$$f(x) = 0 \quad \text{for} \quad |x| > N, \qquad N < \infty, \qquad \int_{-N}^{N} f(x)\, dx = 1 \tag{56}$$

Based upon equation (54), one arrives at the following equation instead of equation (38):

$$\dot{x} = \gamma \sin^k \left[\frac{\sqrt{\omega}}{\alpha} p(x) \right] \sin \omega t, \qquad p(x) = 2 \int_{-N}^{x} f(\xi)\, d\xi - 1 \tag{57}$$

Indeed, introducing a new variable [compare with equation (41)]

$$y = p(x), \qquad y(-N) = -1, \qquad y(N) = 1$$

one obtains instead of equation (54)

$$f(x) = \frac{1}{2}|y'| = \frac{1}{2}\frac{dp}{dx} \tag{58}$$

Turning to an n-dimensional dynamical system, we confine ourselves to a special form:

$$\dot{x}_i = \gamma_i \sin^k \left[\frac{\sqrt{\omega}}{\alpha} p_i(y_i) \right] \sin \omega t \tag{59}$$

where

$$y_i = \sum_{j=1}^{n} T_{ij} x_j, \qquad T_{ij} = \text{const} \tag{60}$$

It will be assumed that

$$\frac{dp_i}{dy_i} \begin{cases} >0 & \text{for } |y_1| < N_i \\ =0 & \text{for } |y_i| > N_i \end{cases} \qquad N_i < \infty \tag{61}$$

and T_{ij} forms a symmetric positive-definite matrix. The last property provides stability (if $\sin \omega t < 0$) or instability (if $\sin \omega t > 0$) of the system (59) at the terminal equilibrium point.

Based upon this, one concludes that the system (59) will be locally stable or locally unstable depending upon the sign of $\sin \omega t$, and that synchronizes the conversions of terminal attractors into terminal repellers and vice versa.

Exploiting the result (58), one obtains that the solution to equation (59) has the following density function in terms of the variables y_i:

$$f(y_1, \ldots, y_n) = \prod_{i=1}^{n} p_i'(y_i), \qquad p' = \frac{dp}{dy} \tag{62}$$

In terms of the variables x_i, the joint density of the solution is

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_i'(y_i) \cdot \det|T_{ij}| \tag{63}$$

where y_i is expressed via x_i by equation (60).

3.4. Examples

Example 1. We start with the following problem: Find a dynamical system whose solution is attracted to a stochastic process with the normal density

$$f(x) = z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \tag{64}$$

where μ and σ are the mean and the standard deviation, respectively, and $z(y)$ is the standard normal density function. In order to apply (58), first (64) should be modified, since it does not satisfy the restriction (56).

We will introduce a truncated standard normal density function:

$$\tilde{z}(y)\begin{cases} >z(y) & \text{if } |y|<N \\ =0 & \text{if } |y|>N \end{cases} \qquad N<\infty \tag{65}$$

Then, with reference to equation (57), one obtains

$$\dot{x}=\gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha}\,\widetilde{\text{erf}}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]\sin\omega t, \qquad \widetilde{\text{erf}}(y)=\frac{2}{\sqrt{\pi}}\int_0^y \tilde{z}(u)\,du \tag{66}$$

Thus, equation (66) represent a dynamical system whose solution is attracted to a stochastic process with the density function (65). For a sufficiently large N it will be close to a Gaussian process with μ and σ as the mean and the standard deviation, respectively.

Example 2. Let us assume now that the density $f(x)$ of a sought stochastic process is characterized by $\mu=\mu_0$, $\sigma=\mu_1$, and higher central moments μ_r. Utilizing the Gram–Charlier series expansion

$$f(x)=\frac{1}{\sigma}\sum_{r=0}^{\infty} c_r\tilde{z}^{(r)}\left(\frac{x-\mu}{\sigma}\right) \tag{67}$$

where

$$\begin{aligned} &c_0=1, \qquad c_1=c_2=0, \qquad c_3=-\frac{1}{3!}\mu_3, \qquad c_4=\frac{1}{4!}(\mu_4-3) \\ &c_5=-\frac{1}{5!}(\mu_5-10\mu_6), \qquad c_6=\frac{1}{6!}(\mu_6-15\mu_4+30), \qquad \text{etc.} \end{aligned} \tag{68}$$

$$\tilde{z}^{(r)}=\frac{d^r\tilde{z}(y)}{dy^r} \tag{69}$$

and applying equation (57), one obtains

$$\dot{x}=\gamma\sin^k\left\{\frac{\sqrt{\omega}}{\alpha}\left[\widetilde{\text{erf}}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)+\sum_{r=3}^{\infty}c_r\tilde{z}^{r-1}\left(\frac{x-\mu}{\sigma}\right)\right\}\sin\omega t \tag{70}$$

Hence, the solution to the dynamical system (70) is attracted to a stochastic process whose density function is characterized by the moments μ_r.

Example 3. In this example we will pose the following problem: Find a dynamical system whose solutions $x_i(t)$ are attracted to a stochastic process characterized by the column of means and the matrix of moments

$$Mx_i=\mu_i, \qquad \sigma_{ij}=M(x_i-\mu_i)(x_j-\mu_j), \qquad i,j=1,2,\ldots,n \tag{71}$$

First one can find such an orthogonal transformation

$$y_i = \eta_i + \sum_{j=1}^{n} T_{ij}(x_j - \mu_j) \tag{72}$$

that

$$My_i = \eta_i = 0, \qquad \sigma'_{ik} = \sum_{j=1}^{n} \sum_{l=1}^{n} \sigma_{jl} T_{ij} T_{kl} = \delta_{ik} = \begin{cases} 1 & \text{if} \quad i = k \\ 0 & \text{if} \quad i \neq k \end{cases} \tag{73}$$

where y_i are noncorrelated standard normally distributed variables.

Combining equations (59), (60), and (66), one obtains

$$\dot{x}_i = \gamma_i \sin^k\left[\frac{\sqrt{\omega}}{\alpha} \,\widetilde{\text{erf}}\left(\frac{y_i}{\sqrt{2}}\right)\right] \sin \omega t, \qquad y_i = \sum_{j=1}^{n} T_{ij}(x_j - \mu_j) \tag{74}$$

Some comments concerning the stability of equations (74) should be made. Since T_{ij} is an orthogonal matrix, the real parts of the eigenvalues of T_{ij} are

$$\text{Re}\,\lambda_i = \cos \varphi_i > 0 \qquad \text{for} \quad 0 \leq \varphi < \pi/2 \tag{75}$$

where φ_i are the angles of rotation of the coordinate axes, and since

$$\frac{d}{dy_i}\,\widetilde{\text{erf}}(y_i) > 0 \qquad \text{for} \quad |y| < N_i \tag{76}$$

i.e., the condition (61) is satisfied, equations (74) linearized with respect to their equilibrium points have eigenvalues whose real parts are all positive (if $\sin \omega t > 0$) or negative (if $\sin \omega t < 0$), and that synchronizes conversions from terminal attractors to terminal repellers and vice versa.

Thus, the solution to the dynamical system is attracted to a stochastic process with prescribed probabilistic structure (71) if the initial conditions are within the basin of attraction: $|y_i| < N_i$.

4. STOCHASTIC ATTRACTOR AS A TOOL FOR GENERALIZATION

Random activity in the human brain is a subject of discussion in many recent publications (Harth, 1993; Osovets, 1983). The interest in the problem was promoted by the discovery of strange attractors. This discovery provided a phenomenological framework for understanding electroencephalogram data in regimes of multiperiodic and random signals generated by the brain. An understanding of the role of such random states in the logical structure of human brain activity would significantly contribute not only to brain science, but also to the theory of advanced computing based upon artificial neural networks. In this section, based upon properties of terminal neurodynamics discussed above, we propose a phenomenological approach to the problem: we demonstrate that a stochastic attractor incorporated in neural net models can represent a *class* of patterns, i.e., a collection of all those and only those patterns to which a certain concept applies. Formation of such a class is associated with higher-level cognitive processes (generalization). This generalization is based upon a set of unrelated patterns represented by static attractors and associated with the domain of a lower level of brain activity (perception, memory). Since a transition from a set of unrelated static attractors to the unique stochastic attractor releases many synaptic interconnections between the neurons, the formation of a class of patterns can be "motivated" by a tendency to minimize the number of such interconnections at the expense of omitting some insignificant features of individual patterns.

Let us first consider a deterministic dissipative nonlinear dynamical system modeled by a coupled set of first-order differential equations of the form

$$\dot{x}_i = V_i(x_j, T_{ij}), \qquad i, j = 1, 2, \ldots, n \tag{77}$$

in which x_i is an n-dimensional vector function of time representing the neuron activity, and T_{ij} is a constant matrix whose elements represent synaptic interconnections between the neurons.

The most important characteristic of the neurodynamic systems (77) is that they are dissipative, i.e., their motions, on the average, contract phase space volumes onto attractors of lower dimensionality than the original space.

So far only point attractors have been utilized in the logical structure of neural network performance: they represent stored vectors (patterns, computational objects, rules). The idea of storing patterns as point attractors of neurodynamics implies that initial configurations of neurons in some neighborhood of a memory state will be attracted to it. Hence, a point attractor (or a set of point attractors) is a paradigm for neural net performance based upon the phenomenology of nonlinear dynamical systems. This performance is associated with the domain of lower-level brain activity such as perception and memory.

It is easily verifiable that a set of point attractors imposes certain constraints upon the synaptic coefficients T_{ij}. Indeed, for a set of m fixed points $\tilde{x}_i^k$ $(k = 1, 2, \ldots, m)$ one obtains $m \times n$ constraints following from (77):

$$0 = V_i(\tilde{x}_j^k, T_{ij}), \qquad i, j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, m \tag{78}$$

In order to provide stability of the fixed points $\tilde{x}_j^k$, the synaptic coefficients must also satisfy the following $m \times n$ inequalities:

$$\operatorname{Re} \lambda_i^k < 0, \qquad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, m \tag{79}$$

in which λ_i^k are the eigenvalues of the matrices $\|\partial f_i / \partial x_j\|$ at the fixed points $\tilde{x}_i^k$.

How can a neural network minimize the number of interconnections T_{ij} without a significant loss of the quality of a prescribed performance?

Let us assume that the vectors $\tilde{x}_j^k$ have some characteristics in common, for instance, their ends are located on the same circle of a radius r_0, i.e. (after proper choice of coordinates)

$$\sum_{i=1}^{2} (\tilde{x}_i^k)^2 = r_0^2, \qquad k = 1, 2, \ldots, m \tag{80}$$

If for the patterns represented by the vectors $\tilde{x}_i^k$ the property (80) is much more important than their angular coordinates θ^k ($\theta^{k1} \neq \theta^{k2}$ if $k_1 \neq k_2$), then it is "reasonable" for the neural net to store the circle $r = r_0$ instead of storing m point attractors with at least $2 \times m$ synaptic coefficients T_{ij}. Indeed, in this case the neural net can "afford" to eliminate unnecessary synaptic coefficients by reducing its structure to the simplest form:

$$\dot{r} = r(r - r_0)(r - 2r_0), \qquad \dot{\theta} = \omega = \text{const} \tag{81}$$

Equations (81) have a periodic attractor

$$r = r_0, \qquad \theta = \omega t \tag{82}$$

which generates harmonic oscillations with frequency ω. But what is the role of these oscillations in the logical structure of neural net performance? The transition to the form (81) can be interpreted as a generalization procedure in the course of which a collection of unrelated vectors $\tilde{x}_i^k$ is united into a class of vectors whose lengths are equal to r_0. Hence, in terms of symbolic logic, the circle $r = r_0$ is a logical form for the class of vectors to which the concept (80) applies. In other words, the oscillations (82) represent a higher-level cognitive process associated with generalization and abstraction. During these processes, the point describing the motion of (81) in the phase space will visit all those and only those vectors whose lengths are equal to r_0; thereby the neural network "keeps in mind" all the members of the class.

Suppose that a bounded set of isolated point attractors which can be united in a class occupies a more complex subspace of the phase space, i.e., instead of the circle (80) the concept defining the class is

$$\Phi(\tilde{x}_1^k, \tilde{x}_2^k, \ldots, \tilde{x}_n^k) = r, \qquad k = 1, 2, \ldots, m \tag{83}$$

Then the formation of the class will be effected by storing a surface

$$\Phi(x_1, x_2, \ldots, x_n) = r \tag{84}$$

as a limit set of the neurodynamics, while all the synaptic coefficients T_{ij} which impose constraints on the velocities along the surface (84) will be eliminated.

The character of the motion on the limit set depends upon the properties of the surface (84). If (by proper choice of coordinates) this surface can be approximated by a topological product of $n-1$ circles [i.e., by an $(n-1)$-dimensional torus], then the motion is quasiperiodic: it generates oscillations with frequencies which are dense in the real. If the surface (84) is more complex and is characterized by a fractial dimension, the motion on such a limit set must be chaotic: it generates oscillations with continuous spectrum. In both cases the motion is ergodic: the point describing the motion in the phase space sooner or later will visit all the points of the limit set, i.e., the neural net will "keep in mind" all the members of the class.

Thus, it can be concluded that artificial neural networks are capable of performing high-level cognitive processes such as formation of classes of patterns, i.e., formation of new logical forms based upon generalization procedures. In terms of the phenomenology of nonlinear dynamics these new logical forms are represented by limit sets which are more complex than point attractors, i.e., by periodic or chaotic attractors. It is shown that formation of classes is accompanied by elimination of a large number of extra synaptic interconnections. This means that these high-level cognitive processes increase the capacity of the neural network. The procedure of formation of classes can be initiated by a tendency of the neural network to minimize the number (or the total strength) of the synaptic interconnections without a significant loss of the quality of prescribed performance; such a tendency can be incorporated into the learning dynamics which controls these interconnections (Zak, 1989*b*).

In addition, the phenomenological approach presented above leads to a possible explanation of random activity of the human brain; it suggests that this activity represents the high-level cognitive processes such as generalization and abstraction.

Turning to the terminal neurodynamics represented by equations (59) and (60), one can view a stochastic attractor as a more universal tool for generalization. Indeed, in contradistinction to chaotic attractors of deterministic dynamics, stochastic attractors can provide any arbitrarily prescribed probability distributions (63) by an appropriate choice of (fully deterministic!) synaptic weights T_{ij}.

The information stored in a stochastic attractor can be measured by the entropy H via the probabilistic structure of this attractor:

$$H(X_1, \dots, X_2, \dots, X_n) = -\sum_{X_1} - \cdots \sum_{X_n} f(x_1, \dots, x_n) \log f(x_1, \dots, x_n) \tag{85}$$

where the joint density $f(x_1, \dots, x_n)$ is uniquely defined by the synaptic weights T_{ij} by means of equation (63).

For instance, the information stored by the dynamical system (66) is measured by the entropy:

$$H = \log_2[(2\pi e\sigma^2)^{1/2}] \tag{86}$$

since this system has a stochastic attractor with normal density distribution (64).

As shown in Zak (1993*b*), terminal neurodynamic systems can have several, or even infinitely many, stochastic attractors. For instance, the dynamical system

$$\dot{x}_1 = \gamma_1 \sin^k[\sqrt{\omega} \sin(x_1 + x_2)] \sin \omega t \tag{87}$$

$$\dot{x}_2 = \gamma_2 \sin^k[\sqrt{\omega} \sin(x_1 + x_2)] \sin \omega t \tag{88}$$

has stochastic attractors with the following densities:

$$f(x_1, x_2) = 0.5|\cos(x_1 + x_2) \cos(x_1 - x_2)| \tag{89}$$

$$\frac{\pi m_1}{2} < x_1 + x_2 < \frac{\pi(m_1 + 2)}{2}, \qquad \frac{\pi m_2}{2} < x_1 - x_2 < \frac{\pi(m_2 + 2)}{2} \tag{90}$$

$$m_1 = \dots, -7, -3, 1, 5, 9, \text{etc.}; \qquad m_2 = \dots, -5, -1, 3, 5, 7, \dots. \tag{91}$$

The solution (89) represents a stationary stochastic process which attracts all the solutions with initial conditions within the area (90). Each pair m_1 and m_2 from the sequences (91) defines a corresponding stochastic attractor with the joint density (89).

Hence, the dynamical system (87), (88) is capable of discrimination between different stochastic patterns, and therefore it performs pattern recognition on the level of classes.

5. COLLECTIVE BRAIN PARADIGM

In this section the usefulness of terminal neurodynamics, and in particular of a new dynamical phenomenon—stochastic attractor—will be demonstrated by simulating a paradigm of a collective brain.

5.1. General Remarks

The concept of the collective brain has appeared recently as a subject of intensive scientific discussions from the theological, biological, ecological, social, and mathematical viewpoints (Huberman, 1989; Seeley and Levien, 1988). It can be introduced as a set of simple units of intelligence (say, neurons) which can communicate by exchange of information without explicit global control. The objectives of each unit may be partly compatible and partly contradictory, i.e., the units can cooperate or compete. The exchanging information may be at times inconsistent, often imperfect, nondeterministic, and delayed. Nevertheless, observations of working insect colonies, social systems, and scientific communities suggest that such collectives of single units appear to be very successful in achieving global objectives, as well as in learning, memorizing, generalizing, and predicting, due to their flexibility, adaptability to environmental changes, and creativity.

In this section collective activities of a set of units of intelligence will be represented by a dynamical system which imposes upon its variables different types of nonrigid constraints such as probabilistic correlations via the joint density. It is reasonable to assume that these probabilistic correlations are learned during a long-term period of performing collective tasks. Due to such correlations, each unit can predict (at least in terms of expectations) the values of parameters characterizing the activities of its neighbors if the direct exchange of information is not available. Therefore, a set of units of intelligence possessing a "knowledge base" in the form of a joint density function is capable of performing collective purposeful tasks in the course of which the lack of information about current states of units is compensated by the predicted values characterizing these states. This means that actually in the collective brain global control is replaced by the probabilistic correlations between the units stored in the joint density functions.

Since classical dynamics can offer only fully deterministic constraints between the variables, we will turn to its terminal version discussed in the previous sections. Based upon the stochastic attractor phenomenon as a paradigm, we will develop a dynamical system whose solutions are stochastic processes with prescribed joint density. Such a dynamical system introduces more sophisticated relationships between its variables which resemble those in biological or social systems, and it can represent a mathematical model for the knowledge base of the collective brain.

5.2. Model of Collective Brain

Let us first turn to an example and consider a basketball team. One of the most significant properties of success in a game is the ability of each player to predict the actions of his or her partners even if they are out of that player's visual field. Obviously, such an ability should be developed in the course of training experience. Hence, the collective brain can be introduced as a set of simple units of intelligence which achieve the objective of the team without explicit global control; actions of the units are coordinated by the ability to predict the values of parameters characterizing the activities of their partners based upon the knowledge acquired and stored during long-time experience of performing similar collective tasks.

We will start with the mathematical formulation of the collective brain for a set of two units considered in the previous section and described by equations (87), (88). As shown there, this system has a random solution which eventually approaches a stationary stochastic process with the joint probability density (89). For further analysis we will take $m_1 = 1$, $m_2 = -1$ from equation (91).

As follows from the solution (89), one can find the probability density characterizing the behavior of one unit (say x_1) given the behavior of another, i.e., x_2.

Let us assume now that the unit x_1 does not have information about the behavior of the unit x_2. Then the unit x_1 will turn to the solution (89), which is supposed to be stored in its memory. From this solution, the conditional expectation of x_2 given x_1 can be found as

$$E(x_2|x_1 = x_1) = \int_{-\infty}^{\infty} x_2 f_2(x_2|x_1)\, dx_2 = \tilde{x}_2 \tag{92}$$

in which the conditional density

$$f_2(x_2|x_1) = \frac{f(x_1, x_2)}{f_1(x_1)} \tag{93}$$

where the marginal density is

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 \tag{94}$$

Actually the integration with respect to x_2 in (53) and (55) is taken over the square $ABCD$ in Fig. 4.

Substituting (94) and (93) into (92), one obtains

$$\tilde{x}_2 \cong \pi/2, \qquad 0 < x_2 < \pi \tag{95}$$

Similarly,

$$\tilde{x}_1 \cong \pi/2, \qquad 0 < x_2 < \pi \tag{96}$$

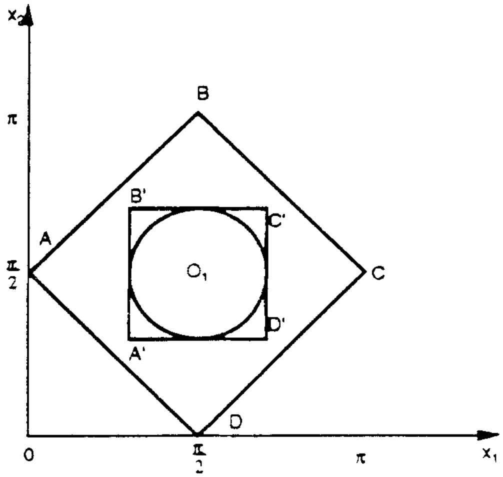

Fig. 4. Collective brain with fuzzy objective.

Clearly this result should be expected based upon the symmetry of the square *ABCD* and the joint probability density (89) with respect to the lines (95) and (96).

It should be noticed that in the general case

$$\tilde{x}_2 = \tilde{x}_2(x_1) \qquad \text{and} \qquad \tilde{x}_1 = \tilde{x}_1(x_2) \tag{97}$$

Substituting (95) and (96) into (87) and (88), respectively, one obtains

$$\dot{x}_1 = \gamma_1 \sin^k\left[\sqrt{\omega}\sin\left(x_1 + \frac{\pi}{2}\right)\right]\sin \omega t \tag{98}$$

$$\dot{x}_2 = \gamma_1 \sin^k\left[\sqrt{\omega}\sin\left(\frac{\pi}{2} - x_2\right)\right]\sin \omega t \tag{99}$$

The system (98), (99) respresents the collective brain derived from the original system (87), (88).

Both equations (98) and (99) are self-contained: they are formally independent since the actual contribution of the other unit is replaced by the "memory" of its typical performance during previous (similar) collective tasks. This memory is extracted from the joint probability density (89) in the form of the conditional expectations (95), (96).

The probability densities for the performances of x_1 and x_2 in the collective brain are

$$\varphi_1(x_1) = \begin{cases} \frac{1}{2}|\cos(x_1 + \pi/2)|, & 0 \le x_1 \le \pi \\ 0, & \text{otherwise} \end{cases} \tag{100}$$

$$\varphi_2(x_2) = \begin{cases} \frac{1}{2}|\cos(\pi/2 - x_1)|, & 0 \le x_2 \le \pi \\ 0, & \text{otherwise} \end{cases} \tag{101}$$

and therefore their joint probability density is

$$\varphi \circ (x_1, x_2) = \begin{cases} \frac{1}{4}|\cos(x_1 + \pi/2)\cos(\pi/2 - x_2)|, & 0 \le x_1, x_2 \le \pi \\ 0, & \text{otherwise} \end{cases} \tag{102}$$

Obviously, (102) is different from (89), and therefore, strictly speaking, the performance (100), (101) of the units x_1 and x_2 in the collective brain (98), (99) is different from their original performance (89) when all the information about the other unit is available. However, this difference should not be significant if a new task belongs to the same class for which these units were trained.

The dynamical paradigm described above can be easily generalized to an n-dimensional system,

$$\dot{x}_i = \gamma_i \sin^k\left[\frac{\sqrt{\omega}}{\alpha} p_i(y_i)\right] \sin \omega t, \qquad x_i = \sum_{j=1}^{n} T_{ij}, \qquad x_j, t_{ij} = \text{const} \tag{103}$$

while

$$p_i' = \frac{dp_i}{dv_i} \begin{cases} > 0 & \text{for } |y_i| < N_i \\ = 0 & \text{for } |y_i| > N_i \end{cases} \qquad N_i < \infty \tag{104}$$

As shown above, the solution to (103) is random, and it eventually approaches a stationary stochastic process with the joint probability density (63).

Following the previous example, we will assume that the ith unit x_i has actual input only from the units $x_{k(i)}$, $k < n$, so that the rest of the inputs should be predicted based upon the joint probability density (63) which was learned by each unit in the course of the previous collective tasks. Now instead of (103) one can introduce the collective brain:

$$\dot{x}_i = \gamma_i \sin^k\left[\frac{\sqrt{\omega}}{\alpha} p_i(\overset{*}{y}_i)\right] \sin \omega t \tag{105}$$

$$\overset{*}{y}_i = \sum_{j=1}^{k(i)} T_{ij} y_j + E\left(\sum_{k(i)+1}^{n} T_{ij} y_i | x_i, \ldots, x_{k(i)}\right) \tag{106}$$

Here the unavailable input from the units $x_{k(i)1}, \ldots, x_n$ is replaced by their conditional expectation given $x_1, \ldots, x_{k(i)}$. As in the two-dimensional case considered above, this expectation is uniquely defined by the joint probability distribution (63). In the extreme case $k = n - 1$, i.e., when the actual information from other units is not available, equation (106) reduces to the following:

$$\overset{*}{v}_i = v_i + E\left(\sum_{j \neq 1}^{n} T_{ij} v_j \middle| u_i\right) \tag{107}$$

while the conditional expectation in (107) depends only on x_i. This means that all the units in the collective brain (105), (106) are formally independent. But as in the example considered above, their performances are coupled by the "memories" of previous collective tasks stored in the form of conditional expectations.

It should be stressed that the main advantage of the collective brain is in its universality: it can perform a purposeful activity without global control, i.e., with only partial exchange of information between its units. That is why the collective brain can model many collective tasks which occur in real life. Obviously the new tasks are supposed to belong to the same class for which the units were trained. In other words, too many novelties in a new task may reduce the effectiveness of the collective brain.

5.3. Collective Brain with Fuzzy Objective

So far we have been concerned with the structure of the model of collective brain regardless of the objective of its performance. In this section we will discuss the collective brain with objectives for its performance. Usually the objective of a performance is reduced to the minimization of a function or a functional subject to some constraints. In this way the problem has at least a rigorous mathematical formulation, although its solution may not be simple. However, in most practical, real-life problems (for instance, in operations research) the information about the objective is vague, imprecise, and uncertain. In mathematical terms it means that the analytical structure of the function (or the functional) to be minimized is not available. Clearly, this kind of problem is the best "match" for the collective brain, whose low precision is compensated by a high degree of universality. Actually the main motivation for the development of the mathematical model of collective brain was its ability to perform in a more "human" way when rigid rules are replaced by the ordering of multichoice actions with respect to "preference."

In this section we will discuss fuzzy objectives, which are given by a system of inequalities. We will start with a two-dimensional example, and turn to (87), (88). However, now we will assume that $\omega \neq \text{const}$, and in particular

$$\omega = \begin{cases} 0 & \text{if} \quad \theta < 0 \\ \omega_0 & \text{if} \quad \theta > 0 \end{cases} \qquad \omega_0 > 0 \tag{108}$$

The inequality (108) can be implemented by the following dynamical equation (Zak, 1993*a*):

$$\dot{\omega} = [\omega(\omega_0 - \omega)]^{1/3}\theta - \delta^2\left(\omega - \frac{1}{2}\right), \qquad \delta^2 \to 0 \tag{109}$$

(This kind of dynamical system will be discussed in the next sections.) Here θ can be an arbitrary function of x_1 and x_2, for instance

$$\theta = (x_1 - \alpha_1)^2 + (x_2 - \alpha_2)^2 - a^2 \tag{110}$$

where α_1, α_2, and a are given constants.

As follows from equation (110), all the states of the system (87), (88), and (108) which are inside of the circle

$$(x_1 - \alpha_1)^2 + (x_2 - \alpha_2)^2 = a^2 \tag{111}$$

will correspond to its equilibria, since then $\theta < 0$ and $\omega = 0$. But since the solution to these equations is random, it can approach an equilibrium at different points inside of this circle, i.e., the final equilibrium point will be characterized by some uncertainty.

It is worth emphasizing that this uncertainty is not explicitly imposed by any rigid rule: it is generated by the dynamical system itself as a result of the randomness of its behavior, and the fuzziness of the objective function

$$\theta < 0 \tag{112}$$

Obviously the circle (111) (or at least part) must be inside of the square *ABCD* (Fig. 4). For that one can take, for instance,

$$\alpha_1 = \alpha_2 = \pi/2 \tag{113}$$

The objectives of the units u_1 and u_2 are not necessarily identical, but they must be compatible. For instance, instead of (87), (88), (108), and (109) one can have

$$\dot{x}_1 = \gamma_1 \sin^k[\sqrt{\omega_1} \sin(x_1 + x_2)] \sin \omega_1 t \tag{114}$$

$$\dot{x}_2 = \gamma_2 \sin^k[\sqrt{\omega_2} \sin(x_1 - x_2)] \sin \omega_2 t \tag{115}$$

However, now we will assume that $\omega_i \neq \text{const}$, and in particular

$$\omega_i = \begin{cases} 0 & \text{if} \quad \theta_i < 0 \\ \omega_0 & \text{if} \quad \theta_i > 0 \end{cases} \qquad i = 1, 2 \tag{116}$$

while

$$\theta_1 = \left(x_1 - \frac{\pi}{2}\right)^2 + \left(x_2 - \frac{\pi}{2}\right)^2 - a^2 \tag{117}$$

$$\theta_2 = (a - l)^2 - \left(x_1 - \frac{\pi}{2}\right)^2 - \left(x_2 - \frac{\pi}{2}\right)^2 \tag{118}$$

In this case the area of the equilibria of the system (114)–(118) is within the ring of width l:

$$\left(x_1 - \frac{\pi}{2}\right)^2 + \left(x_2 - \frac{\pi}{2}\right)^2 = a^2, \qquad \left(x_1 - \frac{\pi}{2}\right)^2 + \left(x_2 - \frac{\pi}{2}\right)^2 = (a - l)^2 \quad (119)$$

and therefore the solution can approach any of its points.

We can give the following interpretation of the performance of this system. Let us assume that the units u_1 and u_2 were trained to perform a certain class of collective tasks which led to formation of the soft constraints

$$T_{11} = 1, \qquad T_{22} = -1, \qquad T_{12} = T_{21} = 1 \quad (120)$$

so that the random behavior of x_1 and x_2 eventually approaches a stationary stochastic process with the joint probability density (89). Let us also assume that a new task (which belongs to the same class for which the system was previously trained) is to optimize some process which depends upon the values x_1 and x_2. Here we will be interested in the cases when this dependence is given in an uncertain, imprecise way, which is typical for the problems in decision-making processes. The simplest mathematical formalization of such a dependence is expressed by the "yes-or-no" relationship with respect to a certain discrimination surface

$$\theta_i(x_1, x_2) = 0, \qquad i = 1, 2, \ldots, n \quad (121)$$

In particular, the values x_1 and x_2 are optimal if

$$\theta_i < 0 \quad (122)$$

and nonoptimal if

$$\theta_i > 0 \quad (123)$$

As follows from equations (114)–(116), in the case (122) the dynamical system will remain in equilibrium, while in the case (123) this system will evolve until it approaches the area (122).

In general, the criterion of the optimality may change in time,

$$\theta_i(x_1, x_2, t) = 0 \quad (124)$$

while θ_i is a slow function of time in the sense that

$$\left|\frac{\partial \theta}{\partial t}\right| \ll \theta\omega \quad (125)$$

If the system (114)–(116) was in an equilibrium, eventually it will be activated again when the inequality (122) changes to (123), and then it will evolve until a new equilibrium in the area (122) will be approached again, etc.

In the most general case the parameters which define the criterion of optimality may themselves be governed by a dynamical process which is controlled by the original dynamical system, for instance,

$$\dot{a}_1 = \lambda_1 \sin^k[\sqrt{\Omega_1} \sin(a_1 + a_2)] \sin \Omega_1 t \tag{126}$$

$$\dot{a}_2 = \lambda_2 \sin^k[\sqrt{\Omega_2} \sin(a_1 + a_2)] \sin \Omega_1 t \tag{127}$$

$$\omega_i = \begin{cases} 0 & \text{if} \quad \Gamma_i < 0 \\ \omega_0 & \text{if} \quad \Gamma_i > 0 \end{cases} \tag{128}$$

θ_i in (116) and Γ_i in (128) are expressed as

$$\theta_i = \theta_i(x_1, x_2, a_1, a_2), \qquad \Gamma_i = \Gamma_i(a_1, a_2, x_1, x_2) \tag{129}$$

The relationship between equations (114)–(116) and (126)–(129) represents a dynamical game which ends when simultaneously

$$\theta_i < 0 \qquad \text{and} \qquad \Gamma_i < 0 \tag{130}$$

Hence the equilibrium values of x_1, x_2, a_1, and a_2 are given by the intersections of the inequalities (129), and they densely fill up a part of the space x_1, x_2, a_1, a_2. Despite the fact that the dynamical behavior of the parameters x_i, a_i is characterized by uncertainties (coming from the randomness of the solutions and the fuzziness of the optimality criteria), the end of the game, i.e., the stationary values of x_i and a_i, can be predicted in probabilistic terms since these values should belong to the stochastic attractor of the system (114)–(116), (126)–(129) whose joint probability density is uniquely defined by the synaptic interconnections (120).

Let us now return to the model of the collective brain, and start with the dynamical system (113)–(115). In reducing this system to the model of the collective brain, one should assume again that each unit does not have explicit information about another, and therefore, the values of x_2 in (114) and the values of x_1 in (115) must be replaced by the conditional expectations $\tilde{x}_2$ and $\tilde{x}_1$, respectively [see (95) and (96)]. But in addition, one has to introduce different rhythms ω_1 and ω_2 which are controlled by different functions θ_1 and θ_2, respectively:

$$\dot{x}_1 = \lambda_1 \sin^k\left[\sqrt{\omega} \sin\left(x_1 + \frac{\pi}{2}\right)\right] \sin \omega_1 t \tag{131}$$

$$\dot{x}_2 = \lambda_2 \sin^k\left[\sqrt{\omega} \sin\left(\frac{\pi}{2} - x_2\right)\right] \sin \omega_2 t \tag{132}$$

$$\omega_i = \begin{cases} 0 & \text{if} \quad \theta_i < 0 \\ \omega_0 & \text{if} \quad \theta_i > 0 \end{cases} \qquad i = 1, 2 \tag{133}$$

while θ_1 and θ_2 are obtained from equation (110) as a result of replacing x_2 and x_1 by their conditional expectations $\tilde{x}_2$ and $\tilde{x}_1$, respectively:

$$\theta_1 = (x_1 - \alpha_1)^2 + (\pi/2 - x_2)^2 - a^2 \tag{134}$$

$$\theta_2 = (\pi/2 + \alpha_1)^2 + (x_2 - \alpha_2)^2 = a^2 \tag{135}$$

Equations (131), (132) have random solutions which are attracted to stationary stochastic processes with independent probability densities (100) and (101), respectively. The system will approach an equilibrium state if simultaneously

$$\theta_1 < 0 \qquad \text{and} \qquad \theta_2 < 0 \tag{136}$$

For $\alpha_1 = \alpha_2 = \pi/2$, the area of possible equilibria is the square $A'B'C'D'$ of Fig. 4,

$$x_1 = \pi/2 \pm a, \qquad x_2 = \pi/2 \pm a \tag{137}$$

Recalling that for the original system (114) and (115), the area of possible equilibria is inside of the circle of radius a (Fig. 4), one can see that the performance of the collective brain is sufficiently close to the original performance.

It should be stressed that the success of the collective brain performance depends upon how close the new task is to the class of tasks for which the system was trained. This means that the collective brain may fail if the new task has too many "novelties." In the selected example, the training area (see the square $ABCD$ in Fig. 4) is compatible with the objective (the circle with the center 0 in Fig. 4), and therefore the performance of the collective brain (the square $A'B'C'D'$ in Fig. 4) is satisfactory.

6. OPEN SYSTEMS IN TERMINAL NEURODYNAMICS

The neurodynamic models discussed in the previous sections are similar to closed thermodynamic systems in the sense that their entropy can only increase until it reaches its maximum at the stochastic attractor. In this section we will introduce open systems which are maintained in their specific states by an influx of energy, so that information stored in the dynamical system can increase or decrease.

Let us assume that the dynamical system (22) is driven by a vanishingly small input $\varepsilon(t)$:

$$\dot{x} = \gamma \sin^{1/3} \frac{\sqrt{\omega}}{\alpha} x \sin \omega t + \varepsilon(t), \qquad |\varepsilon(t)| \ll \gamma \tag{138}$$

This input can be ignored when $\dot{x} \neq 0$ or when $\dot{x} = 0$, but the system is stable, i.e., $x = \pi\alpha/\sqrt{\omega}$, $3\pi\alpha/\sqrt{\omega}$, etc. However, it becomes significant during the instants of instability when $\dot{x} = 0$ at $x = 0, 2\pi\alpha/\sqrt{\omega}$, etc.

The function $\varepsilon(t) \ll \gamma$ can be associated with a microsystem which controls the neuron behavior through a string of signs (Zak, 1990*b*).

Indeed, actually the only important part in this input is the sign of $\varepsilon(t)$ at the critical points. Consider, for example (138), and suppose that

$$\operatorname{sgn} \varepsilon(t_k) = +, +, -, +, -, -, \text{etc.} \quad \text{at} \quad t_k = \pi k/\omega, \ldots \quad (139)$$
$$k = 0, 1, 2, \ldots$$

The values of $\varepsilon(t)$ in between the critical points are not important since, by our assumption, they are small in comparison to values of the derivative $\dot{x}$ and therefore can be ignored. Hence, the only part of the input $\varepsilon(t)$ which is significant in determining the motion of the neuron (138) is the sign string (139): specification of this string fully determines the dynamics of (138).

Figure 5 demonstrates three different scenarios of motions for the different strings:

$$\varepsilon = \varepsilon_0 \sin \Omega t, \qquad \Omega/\omega = \sqrt{0.2}, \sqrt{2}, \sqrt{20}$$

It should be emphasized that, although these three solutions are bounded and aperiodic, they are fully deterministic in the sense that each of them is uniquely defined by the corresponding initial conditions.

Suppose that

$$\varepsilon(t) = -\varepsilon_0^2 x, \qquad \varepsilon_0^2 \to 0 \qquad (140)$$

i.e.,

$$\dot{x} = \gamma \sin^{1/3} \frac{\sqrt{\omega}}{\alpha} x \sin \omega t - \varepsilon_0^2 x, \qquad \varepsilon_0 \to 0 \qquad (141)$$

It can be verified that the solution to equation (141) will oscillate about the point $x = 0$. Indeed, when the point $x = 0$ becomes a terminal repeller, i.e., when $\sin \omega t > 0$, the solution escapes to the neighboring (right or left) equilibrium point. However, $\dot{x} < 0$ at $x_1 = \pi\alpha/\sqrt{\omega} > 0$ and $\dot{x} > 0$ at $x_1 = -\pi\alpha/\sqrt{\omega}$. Therefore, in both cases the solution returns to the original point $x = 0$. The amplitude and the period of the oscillations about $x = 0$ can be found from (23) and (26), respectively.

However, in contrast to a classical version of equation (141),

$$\dot{x} = -\varepsilon_0^2 x \qquad (142)$$

where $x = 0$ is a static attractor, the same point $x = 0$ is not a static, and

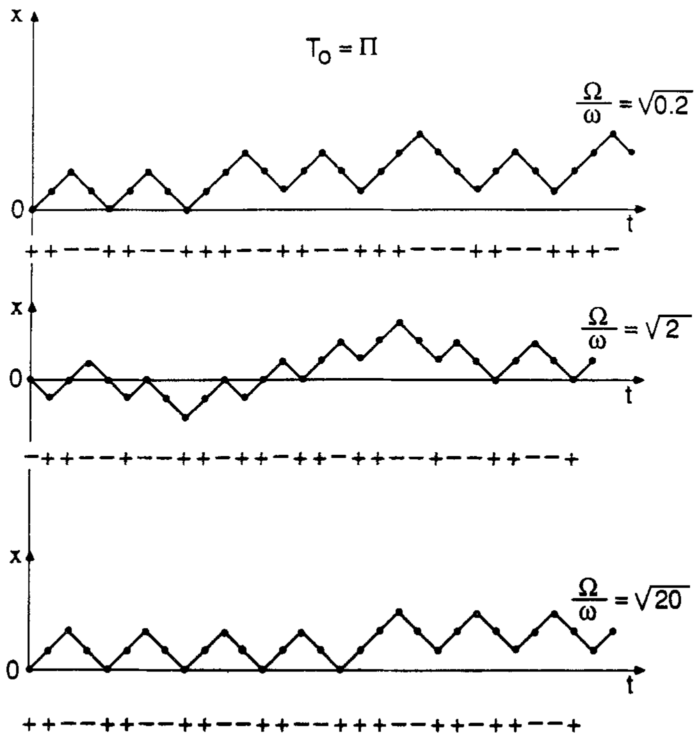

Fig. 5. Temporal patterns and their codes.

not even a periodic, but a stochastic attractor. Indeed, there are several equally probable patterns of oscillations:

$$0, 1, 0, -1, 0; \quad 0, -1, 0, 1, 0; \quad 0, 1, 0, 1, 0; \quad 0, -1, 0, -1, 0, \ldots \tag{143}$$

which can follow each other in an arbitrary order. In probabilistic terms the oscillations can be characterized as $x = 0$ at $t = 2\pi n/\omega$, while

$$\Pr\left\{x\left[t = \frac{(2n+1)\pi}{\omega}\right] = 1\right\} = 0.5$$
$$\Pr\left\{x\left[t = \frac{(2n+1)\pi}{\omega}\right] = -1\right\} = 0.5 \tag{144}$$
$$n = -2, -1, 0, 1, 2, \ldots$$

so the probability of any combinations of the patterns (143) can be found from equation (144).

It should be emphasized that the random stationary process (144) can be considered as a stochastic attractor which is approached by the solution to (141) regardless of initial conditions. But in contradistinction to the stochastic attractors in closed systems considered in the previous sections, where initial disorder could only increase, here the entropy of the initial distribution of x can be higher than that of the attractor (144), i.e., the dynamical system (141) may decrease the initial information. This fundamental difference between closed and open neurodynamic systems is caused by the fact that the evolution of closed systems is driven by pure diffusion [see (31)], while the evolution of open systems is driven by both diffusion and convection. Indeed, the evolution of the probability density of the solution to (141) (for $\sqrt{\omega} \to \infty$) is governed by the Fokker–Planck equation with the drift term:

$$\frac{\partial f}{\partial t} = -\alpha\sqrt{\omega}(q-p)\frac{\partial f}{\partial x} + \frac{1}{2}\pi\alpha^2\frac{\partial^2 f}{\partial x^2} \tag{145}$$

where p and q are the probabilities that the process is directed to the right or to the left, respectively, at each critical point.

Obviously, the last (diffusion) term survives only if

$$(p-1) \sim 1/\sqrt{\omega} \to 0 \tag{146}$$

As follows from equation (141),

$$p - q = \begin{cases} \operatorname{sign} x & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases} \qquad p + q = 1 \tag{147}$$

and therefore equation (145) can be rewritten as

$$\frac{\partial f}{\partial t} = \alpha\sqrt{\omega}\frac{\partial f}{\partial x}\operatorname{sign} x + \frac{1}{2}\pi\alpha^2\frac{\partial^2 f}{\partial x^2}\delta[(p-q)] \tag{148}$$

where δ is the Dirac function.

Let us assume that the initial density of x is uniform:

$$f_0(0) = \begin{cases} c = 1/l & \text{if} \quad |x| < l \\ 0 & \text{otherwise} \end{cases} \tag{149}$$

Then, the solution to equation (148) is

$$f(x) = \begin{cases} f_0[x - (\alpha\sqrt{\omega}\operatorname{sign} x)t] + \dfrac{\alpha\sqrt{\omega t}}{l}\delta(0) & \text{if} \quad 0 < t \leq \dfrac{l}{\alpha\sqrt{\omega}} \\ \delta(0) & \text{if} \quad t > \dfrac{l}{\alpha\sqrt{\omega}} \end{cases} \tag{150}$$

The Dirac function $\delta(0)$ in the solution (150) provides the normalization condition:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{151}$$

The process of the probability density evolution can be interpreted as follows: the initial uniform density (149) "moves" (with the velocity $\pm\alpha\sqrt{\omega}$) toward the point $x = 0$, being absorbed there. At the same time, the Dirac function grows out of the point $x = 0$ in such a way that it compensates the "loss of the area" enveloped by the density $f(x)$. Hence, eventually, the solution approaches the attractor $x = 0$, while the transition period

$$T = \frac{l}{\alpha\sqrt{\omega}} \to 0 \qquad \text{if} \quad \sqrt{\omega} \to \infty \tag{152}$$

However, for

$$t = \frac{l}{\alpha\sqrt{\omega}}$$

the drift term disappears [see equation (147)] and diffusion takes over. For finite $\sqrt{\omega}$ it leads to the random oscillations described by equation (144).

It should be emphasized that the stochastic attractors in open systems [see equation (144)] are different from those in closed systems considered in the previous sections. First, they may have an entropy which is smaller than the initial entropy. Second, the time of approaching these attractors is finite [see (152)]. In order to distinguish these two types of stochastic attractors, those in the open systems were called terminal chaotic attractors, or terminal chaos (Zak, 1991*a*). The similarity between the random oscillations (144) and chaotic attractors in classical dynamics is in the fact that in both cases the phenomena are based upon combined effects of stability and instability. However, in terminal chaos (144) the mechanisms of stability and instability act sequentially: during the first period the neuron is attracted to the point $x = 0$, then it is repelled from it (in one of two possible directions). Second, the time of approaching the center $x = 0$ is finite (due to failure to the Lipschitz condition at $x = 0$). That is why this attractor is terminal. Clearly, terminal chaos is characterized by a well-organized probabilistic structure [see (144)] which simplifies its prediction and control. More general properties of terminal chaos were analyzed in Zak (1991*a*). In order to illustrate some of them, let us consider two independent neurons which have the following microdynamics:

$$\dot{x}_1 = \gamma \sin^{1/3}\frac{\omega}{\alpha} x_1 \sin\omega t - \varepsilon_0^2 x_1, \qquad \delta^2 \to 0 \tag{153}$$

$$\dot{x}_2 = \gamma \sin^{1/3} \frac{\omega}{\alpha} x_2 \sin \omega t + \varepsilon_0^2 x_2 (0.5 - x_2)(x_2 - 1) \tag{154}$$

As shown above, the first neuron performs chaotic behavior with respect to the center $x_1 = 0$, while the second neuron has two centers of terminal chaos: at $x_2 = 0$ and $x_2 = 1$.

Let us introduce now a secondary scale microdynamics with the order of $\delta^4 \to 0$ which couples (153) and (154):

$$\dot{x}_1 = \gamma \sin^{1/3} \frac{\omega}{\alpha} x_1 \sin \omega t - \varepsilon_0^2 x_1 + \varepsilon_0^4 x_2 \tag{155}$$

$$\dot{x}_2 = \gamma \sin^{1/3} \frac{\omega}{\alpha} x_2 \sin \omega t + \varepsilon_0^2 x_2 (0.5 - x_2)(x_2 - 1) + \varepsilon_0^4 x_1 \tag{156}$$

The last terms in these equations are effective only when all the other terms are zero, i.e., at the centers of chaotic attractors $x_1 = 0$, $x_2 = 1$, where the behavior of the neurons is unpredictable. Due to the coupling via the secondary scale microdynamics, the second neuron makes a decision for the first one at $x_1 = 0$ as well as the first neuron makes a decision for the second one at $x_2 = 0$ and $x_2 = 1$. In other words, the chaotic structure of the neurons with the primary microdynamics (153) and (154) reserves room for a match between these two neurons if they work in parallel, without changing their primary microdynamics. In the same way each of these neurons can work in parallel with other neurons while the adjustment between them is carried out by the secondary scale microdynamics due to the chaotic structure of their primary microdynamics.

The secondary order microdynamics does not necessarily eliminate chaos. Indeed, if in (155), (156) $x_1 = 0$ and $x_2 = 0$ simultaneously, then the behavior of the first neuron at $x_1 = 0$ is still unpredictable, and the third-order-scale microdynamics should be incorporated. Hence, one arrives at multiscale microdynamic chains by means of which different neurons are adjusted to each other in their parallel performance on the level of a certain order scale microdynamics. However, the room for such an adjustment is provided by the chaotic structure of microdynamics of uncoupled neurons via the redundancy of available "free" parameters.

Now we take the next step toward neurodynamic complexity and replace equation (141) by the two-scale microdynamics

$$\dot{x} = \gamma \sin^{1/3} \frac{\omega_1}{\alpha_1} x \sin \omega_1 t - \varepsilon_0^2 x \sin \omega_2 t \tag{157}$$

$$\omega_2 \ll \omega_1 \tag{158}$$

During the period

$$t < \pi/\omega_1 \tag{159}$$

the solution to equation (157) behaves exactly as for (141): It has a chaotic attractor at $x = 0$, since

$$\operatorname{sgn} x = \operatorname{sgn}(x \sin \omega_2 t) \qquad \text{at} \quad t < \pi/\omega_2 \tag{160}$$

But in contrast to the latter, the solution to equation (157) is not locked up in this chaotic attractor: eventually it drifts away from the point $x = 0$ since

$$\operatorname{sgn} x = -\operatorname{sgn}(x \sin \omega_2 t) \qquad \text{at} \quad t > \pi/\omega_2 \tag{161}$$

and the scenario of the chaotic oscillations can be the following:

$$0, 1, 0, -1, 0, 1, 2, 1, 2, 3, 4, 3, 4, 5, \ldots$$

or

$$0, 1, 0, -1, 0, -1, -2, -1, -2, -3, -2, -3, -4, \ldots \tag{162}$$

This drift can be bounded if one modifies equation (157) as

$$\dot{x} = \gamma \sin^{1/3} \frac{\omega_1}{\alpha_1} x \sin \omega_1 t - \varepsilon_0^2 x \sin \omega_2 t - \varepsilon_0^4 x \tag{163}$$

The last term representing the second-order microdynamics will return the solution to the chaotic attractor $x = 0$ after the period $t > \pi/\omega_2$; and the scenario of the "double-period" chaotic oscillations will be, for $\omega_1 = 5\omega_2$,

$$0, 1, 0, -1, 0, 1, 2, 3, 4, 5, 4, 3, 2, 1, 0, -1, 0, -1, -2, -3, -4, \ldots$$

or

$$\begin{gathered} 0, 1, 0, -1, -2, -3, -4, -5, -4, -3, -2, -1, \\ 0, 1, 0, -1, -2, -3, -4, \ldots \end{gathered} \tag{164}$$

Hence, despite the fact that the solution to (164) has a more complex temporal structure and a larger number of unpredictable elements, it is still characterized by global coherence: it oscillates chaotically with respect to $x = 0$, while the amplitudes of the oscillations also change chaotically from 1 to 5.

In the same way one can introduce a multiscale dynamics:

$$\dot{x} = \gamma \sin^{1/3} \frac{\omega_1}{\alpha_1} x \sin \omega_1 t + \varepsilon_0^2 x \sin \omega_2 t - \varepsilon_0^4 x \sin \omega_3 t - \varepsilon_0^6, \ldots \tag{165}$$

$$\omega_1 \gg \omega_2 \gg \omega_3, \ldots, \qquad \varepsilon_0 \to 0$$

whose complexity will be proportional to the number of microscales

$\varepsilon_0^2, \varepsilon_0^4, \ldots$ or the number of local times

$$\tilde{t}_k^{(i)} = \frac{\pi}{\omega_i} k, \qquad k = 1, 2, \ldots \tag{166}$$

This multiscale model can be associated with the cascade of intrinsic rhythms which characterize the temporal architecture of mental processes (Geissler, 1987).

Loosely speaking, one can conclude that a neurodynamics with a limited number of local times, or microscales, has limited complexity in the sense that it is locked up in a set of behaviors which belong to a certain class of patterns, and it can escape this class only if the next microlevel (with the corresponding local time) is added.

In this connection an interesting question can be posed: can a microdynamics "improve" itself by producing additional microlevels, or is the number of such levels "genetically" prescribed? So far we do not know the answer.

The structure (165) is the simplest way to increase the complexity of coherent temporal behavior. A more sophisticated approach can be based upon nonlinear effects (Zak, 1991*a*). In order to illustrate this, let us turn to (138) and exploit the following microdynamics:

$$\varepsilon(t) = \varepsilon_0^2 x(x-1.5)(x-4)(4.5-x)(x-5) \tag{167}$$

Here the solution to equation (138) possesses two different terminal chaotic attractors. The first one has its center at $x = 0$ and is characterized by the probabilities (166). The second one has two centers, $x = 4$ and $x = 5$, and is characterized by the probabilities

$$\Pr(x=4) = \Pr(x=5) = 1/3, \qquad \Pr(x=3) = \Pr(x=6) = 1/6 \tag{168}$$

One can verify that their basins of attraction are, respectively,

$$x < 1.5 \qquad \text{and} \qquad x > 1.5 \tag{169}$$

Suppose that the microdynamics (167) includes some external input $f(t)$ which can be interpreted as an outside message:

$$\varepsilon(t) = \varepsilon_0^2[x(x-0.5)(x-4)(4.5-x)(x-5) + f(t)], \qquad \varepsilon_0 \to 0 \tag{170}$$

Then if

$$f(t) > 21 \qquad \text{at} \quad x = 1 \tag{171}$$

the solution which originally was trapped in the first attractor $x = 0$ will escape it and will move to the second attractor with two centers: $x = 4$ and $x = 5$.

Conversely, if

$$f(t) < -13.5 \quad \text{at} \quad x = 3 \tag{172}$$

the solution will return to the first attractor.

Actually the dynamical systems of this type can be regarded as an implementation of a concept of processing of information which includes semantics. This concept is based upon the idea that a meaning can be attributed to a message only if the response of the receiver (a dynamical attractor) is taken into account. In this context the inequalities (171) and (172) can be utilized for evaluation of the "relative importance" of the messages delivered by the outside input $f(t)$.

Let us make the next step toward the complexity of temporal structures and show that under certain conditions the solution can change its attractor; moreover, it can create a new attractor and eliminate the old one. In order to illsutrate this, let us consider the following dynamical system:

$$\dot{x}_1 = \gamma_1 \sin^{1/3} \frac{\omega_1}{\alpha_1} x_1 \sin \omega t - \varepsilon_0^2 x_1 \tag{173}$$

$$\dot{x}_2 = \gamma_2 \sin^{1/3} \frac{\omega}{\alpha_2} x_2 \sin \omega_2 t - \varepsilon_0^2 [x_2(x_1 - 0.5)(x_2 - 1)(x_1 + 0.5)] \tag{174}$$

in which

$$\Delta x_1 = \frac{\pi \alpha_1}{\omega_1} = \frac{\pi \alpha_2}{\omega_2} = \Delta x_2 = 1, \qquad \omega_1 \ll \omega_2 \tag{175}$$

Equation (173) is identical to equation (141): it has a terminal chaotic attractor at $x_1 = 0$ with the pattern of oscillations (143). The solution to (174) has more complex behavior: it has a terminal chaotic attractor $x_2 = 0$ if $x_1 < 0.5$, but this attractor disappears as soon as $x_1 > 0.5$, and the solution approaches a new terminal chaotic attractor $x_2 = 1$. Obviously such a transition has a random nature since the oscillations of x_1 are chaotic. However, the probability of this transition can be found based upon the probability of x_1 given by equation (144).

More general types of microdynamics and their applications to information processing are described in Zak (1991*a*). To conclude this section, we will discuss open systems with spatial coherence.

As shown in Zak (1991*c*), spatial self-organization in actual (physical) space can be achieved by introducing a special type of local interconnection simulating diffusion, dispersion, and convection, while the underlying continuous (in time and space) model is described by field equations in which local interconnections are represented by spatial derivatives of neuron potentials. In this section we will incorporate the diffusion-type local interconnections into the microdynamics assuming that the neuron potential discrete distribution x_i over the indices i can be represented by its continuous analog $x(s)$.

Then the neurodynamics is represented by one partial differential equation:

$$x_t = \gamma \sin^{1/3} \frac{\omega}{\alpha} x \sin \omega t + \delta^2 x_{ss} \tag{176}$$

in which

$$x_t = \frac{\partial x}{\partial t}, \qquad x_{ss} = \frac{\partial^2 x}{\partial s^2}; \qquad \alpha, \gamma = \text{const}$$

while the finite-dimensional version of x_{ss} is

$$x_{ss} \simeq x_{i+1} - 2x_i + x_{i-1} \tag{177}$$

We will select the parameters α and ω such that

$$\frac{|x_{\max}|}{n} \sim \frac{\pi\alpha}{\omega} \tag{178}$$

in which n is the number of neurons. In this case the changes of the neuron potential per unit length and per unit of (local) time are of the same order.

Without loss of generality one can introduce the following initial and boundary conditions:

$$x(s, 0) = 0, \qquad x(0, t) = x(1, t) = 0 \tag{179}$$

Then $x_{ss} = 0$ at $t = 0$, and the solution to equation (176) starts with totally unpredictable behavior: at $t = \pi/\omega$ it approaches the values $\pm\pi\alpha/\omega$, which are randomly distributed over x and therefore the function

$$x = x\left(s, \frac{\pi}{\omega}\right) \tag{180}$$

may have a monstrous configuration. During the next period $\pi/\omega < t < 2\pi/3\omega$ those points at the curve (180) where $x_{ss} > 0$ (and therefore $x < 0$) will go up to $x = 0$, and those points where $x_{ss} < 0$ (and therefore $x > 0$) will go down to $x = 0$. However, as follows from (177), those point x_i for which

$$x_{i-1} = x_i = x_{i+1} \tag{181}$$

will have

$$x_{ss} = 0 \tag{182}$$

The probability of the appearance of such points is

$$\Pr(x_{i-1} = x_i = x_{i+1}) = (0.5)^3 = 0.125 \tag{183}$$

These particular points may return to $x = 0$, but with the equal probability they can move away to the value

$$x_i\left(t = \frac{2\pi}{3\omega}\right) = \begin{cases} 2\pi\alpha/3\omega & \text{if} \quad x_i(t = \pi/\omega) > 0 \\ -2\pi\alpha/3\omega & \text{if} \quad x_i(t = \pi/\omega) < 0 \end{cases} \tag{184}$$

Thus, each point of the curve (180) with the probability 0.9375 will return to the initial configuration $x = 0$, and with the probability 0.0625 will move away from it. Actually the curve

$$x = x(s, 2\pi/3\omega) \tag{185}$$

will be close to $x = 0$ in terms of the mean square distance:

$$\rho = \left[\int_0^1 x^2(s, 2\pi/3\omega)\, ds\right]^{1/2} \to 0 \tag{186}$$

It will coincide with the initial configuration almost everywhere excluding some solitary sharp peaks.

The next step of the evolution will be almost the same as the first step: the solution will approach the configuration with the values $\pm\pi\alpha/\omega$ randomly distributed over x, while the probability that this configuration is identical to (180) has the order $\sim 2^{-n}$ (n is the number of neurons). The solitary peaks where the magnitude of x_{ss} is large will be pushed back toward $x = 0$, etc.

Thus, the solution to equation (176) with the initial and boundary conditions (179) chaotically oscillates about the initial configuration $x = 0$. In other words, it preserves the mean square configuration $x = 0$, while the actual configuration $x(s)$ remains random and unpredictable.

It is worth mentioning that the attraction of the solution to its mean square value is provided by the stability of the solution $x = 0$ to the underlying diffusion equation:

$$x_t = \varepsilon_0^2 x_{ss} \tag{187}$$

subject to the conditions (179), which can be obtained by a superposition of terms with exponentially decaying multipliers:

$$x(s, t) = \sum_{p=1}^{\infty} c_n e^{-(\pi p)^2 \delta^4 t} \sin \pi n s \to 0 \qquad \text{at} \quad t \to \infty \tag{188}$$

Let us introduce now variable boundary conditions [instead of (179)] assuming that they are governed by another dynamical system:

$$x_t(0, t) = \gamma \sin^{1/3}\left[\omega_0 \frac{x(0, t)}{\alpha}\right] \sin \omega t - \delta^2 x(0, t) \tag{189}$$

$$x_t(1, t) = \gamma \sin^{1/3}\left[\omega_0 \frac{x(1, t)}{\alpha}\right] \sin \omega t - \delta^2 x(1, t) \tag{190}$$

where

$$\omega_0 \ll \omega \tag{191}$$

Since the general solution to equation (187) is a family of straight lines,

$$x = c_1 s + c_2 \tag{192}$$

one concludes that the solution to equation (176) with the boundary conditions (189), (190) will oscillate chaotically with respect to different straight lines of the family (192) while the change of these lines is also chaotic [in accord with changes of the boundary conditions following from the dynamics (189), (190)].

Hence, the solution to equation (176) with variable boundary conditions has more unpredictable features, but it still preserves the following property: the mean square solution is always a straight line. In other words, the behavior of the system (176), (189), and (190) represents the general solution to (187) as a mean square of chaotic oscillations, while the behavior of the system (176), (179) represents one of its particular solutions $x = 0$.

If one replaces equation (176) by

$$x_t = \gamma \sin^{1/3} \frac{\omega}{\alpha} x \sin \omega t + \varepsilon_0^2 (x_{ss} - \beta^2 x) \tag{193}$$

then the solution to the system (133), (189), and (190) will oscillate chaotically with respect to the curves of the family

$$x = c_1 \operatorname{sign} \beta s + c_2 \cosh \beta s \tag{194}$$

The stability of these oscillations follows from the stability of the underlying diffusion equation with chain reactions:

$$x_t = \varepsilon_0^2 (x_{ss} - \beta^2 x) \tag{195}$$

whose exponential decay is defined by the eigenvalues

$$\lambda_p = -(\beta^2 + \pi p)^2 \delta^2, \qquad p = 1, 2, \ldots \tag{196}$$

As in the previous case, the choice of the curve from the family (194) is made by the boundary conditions which are controlled by (189), (190), and therefore are oscillating chaotically about the values

$$x(0, t) = 0, \qquad x(1, t) = 0 \tag{197}$$

Again, despite a large number of unpredictable elements, the solution to equation (193) preserves its closeness to curves of the family (194).

7. NEURODYNAMIC MODEL OF INFORMATION FUSION

In this section we will apply open terminal neurodynamic systems for simulating information fusion.

Let us first consider two uncoupled closed systems of the type (66):

$$\dot{x}_1 = \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha} \widetilde{\operatorname{erf}}\left(\frac{x_1}{\sqrt{2}\sigma_1}\right)\right] \sin \omega t \tag{198}$$

$$\dot{x}_2 = \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha} \widetilde{\operatorname{erf}}\left(\frac{x_2}{\sqrt{2}\sigma_2}\right)\right] \sin \omega t \tag{199}$$

Solutions to these equations are random, and they approach stationary stochastic processes with the normal distributions, respectively,

$$f_1(x_1) = \frac{1}{\sigma_1(2\pi)^{1/2}} e^{-x_1^2/2\sigma_1^2}, \qquad f_2(x_2) = \frac{1}{\sigma_1(2\pi)^{1/2}} e^{-x_2^2/2\sigma_2^2} \tag{200}$$

Since x_1 and x_2 are independent, their joint density will be

$$f_{12} = \frac{1}{2\pi\sigma_1\sigma_2} e^{-x_1^2/2\sigma_1 - x_2^2/2\sigma_2} \tag{201}$$

Suppose now that equations (198) and (199) are coupled via the following microdynamics:

$$\varepsilon_1 = \varepsilon_0^2(x_2 - x_1), \qquad \varepsilon_2 = \varepsilon_0^2(x_1 - x_2), \qquad \varepsilon_0^2 \to 0 \tag{202}$$

i.e.,

$$\dot{x}_1 = \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha} \widetilde{\operatorname{erf}}\left(\frac{x_1}{\sqrt{2}\sigma_1}\right)\right] \sin \omega t + \varepsilon_0^2(x_2 - x_1) \tag{203}$$

$$\dot{x}_2 = \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha} \widetilde{\operatorname{erf}}\left(\frac{x_2}{\sqrt{2}\sigma_2}\right)\right] \sin \omega t + \varepsilon_0^2(x_1 - x_2) \tag{204}$$

The global behavior of this system is defined by the behavior of an associated dynamical system:

$$\dot{x}_1 = \varepsilon_0(x_2 - x_1), \qquad \dot{x}_2 = \varepsilon_0(x_1 - x_2) \tag{205}$$

The system (205) has the following set of equilibrium states:

$$x_1 = x_2 \tag{206}$$

All of them are stable since the roots of the corresponding characteristic equation are not positive:

$$\lambda_1 = 0, \qquad \lambda_2 = -2 \tag{207}$$

When the solution to equations (203) and (204) approaches the attractor (206), the dynamical system (203), (204) formally reduces to

(198), (199) with the joint density (201). However, now in (201) one must set

$$x_1 = x_2 = x \tag{208}$$

Substituting (208) into (201) and redefining the constant from the normalization condition, one obtains the following probabilistic property of the solution to (203), (204) at $t \to \infty$:

$$f(x = x_1 = x_2) = \frac{1}{\sigma(2\pi)^{1/2}} e^{-x^2/2\sigma^2} \tag{209}$$

in which

$$\sigma^2 = \frac{1}{1/\sigma_1^2 + 1/\sigma_2^2} < \sigma_1^2, \sigma_2^2 \tag{210}$$

Hence, the neurodynamic system implements fusion of information coming from two independent sources about the same object. As a result of that fusion, the combined information is characterized by an entropy which is smaller than the entropies of each original component of information, i.e., the knowledge about this object is improved.

This paradigm can be generalized to the fusion of n independent sources of information:

$$\begin{aligned} \dot{x}_1 &= \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha} \widetilde{\text{erf}}\left(\frac{x_1}{\sqrt{2}\sigma_1}\right)\right] \sin \omega t + \varepsilon_0^2(x_2 - x_1) \\ \dot{x}_2 &= \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha} \widetilde{\text{erf}}\left(\frac{x_2}{\sqrt{2}\sigma_1}\right)\right] \sin \omega t + \varepsilon_0^2(x_3 - x_2) \\ \dot{x}_n &= \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha} \widetilde{\text{erf}}\left(\frac{x_1}{\sqrt{2}\sigma_1}\right)\right] \sin \omega t + \varepsilon_0^2(x_1 - x_n) \end{aligned} \tag{211}$$

The solution to this system approaches a stationary stochastic process with the probability density (209), where

$$\sigma^2 = \frac{1}{\sum_{i=1}^{n} (1/\sigma_1^2)} \tag{212}$$

In the previous examples, the normal distributions were chosen only for the sake of analytical simplicity: any other distributions can be utilized with the same effect of improvement of the combined information.

The next paradigm of information fusion is associated with pattern recognition. In this section, pattern recognition will be considered as a multistep process. In the first step, the pattern is received at a global level when it can be simulated by a multidimensional stochastic attractor. This attractor represents a class to which the pattern belongs. In the second step, when some additional information becomes available, the original stochastic attractor is replaced by a lower-dimension stochastic attractor which represent a subclass to which the pattern belongs, etc. A chain of such attractors of lower and lower dimensionalities which identifies the pattern with higher and higher accuracy can be implemented by terminal neurodynamics as follows.

Consider a dynamical system (87), (88) which has a stochastic attractor (89), and assume that, as additional information, the regression of x_2 on x_1 is given in the form

$$x_2 = \tfrac{1}{2}x_1 \tag{213}$$

Then, modifying equations (87), (88) as

$$\dot{x}_1 = \gamma_1 \sin^k[\sqrt{\omega}\,\sin(x_1 + x_2)]\,\sin\omega t + \varepsilon_0^2\left(x_2 - \frac{1}{2}x_1\right) \tag{214}$$

$$\dot{x}_2 = \gamma_2 \sin^k[\sqrt{\omega}\,\sin(x_1 + x_2)]\,\sin\omega t + \varepsilon_0^2\left(\frac{1}{2}x_1 - x_2\right) \tag{215}$$

and applying the same line of argumentation as for equations (203), (204), one concludes that the solution to (214) and (215) approaches a one-dimensional attractor which is obtained from (89) as a result of replacing x_2 by its expression from (213):

$$f(x_1) = \begin{cases} c\left|\cos\dfrac{3x_1}{2}\cos\dfrac{x_1}{2}\right|, & \dfrac{\pi}{3} \le x_1 \le \pi \\ 0 & \text{otherwise} \end{cases} \tag{216}$$

and the constant c is found from the normalization condition:

$$c = \frac{1}{\int_{\pi/3}^{\pi} |\cos(3x_1/2)\cos(x_1/2)|\,dx_i} = \frac{3\sqrt{3}}{8} \tag{217}$$

One can verify that the entropy of the attractor (216) is smaller than the entropy of the original attractor (89).

In terms of pattern recognition, the original attractor (89) can be identified with a class of patterns in which each pattern is characterized by the parameters x_1 and x_2. Any combination of these parameters has a certain probability to appear in a particular pattern. Additional information about the dependence between x_1 and x_2 for the pattern to be recognized extracts a subclass of pattern (216) to which this pattern belongs.

In order to illustrate a chain of stochastic attractors, start with the dynamical system (74) for $i = 1, 2, 3$ and introduce the following two-cascade microdynamics:

$$\dot{x}_1 = \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha}\,\widetilde{\text{erf}}\left(\frac{y_1}{\sqrt{2}}\right)\right]\sin\omega t + \varepsilon_0^2(x_2 + x_3 - x_1) \tag{218}$$

$$\dot{x}_2 = \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha}\,\widetilde{\text{erf}}\left(\frac{y_2}{\sqrt{2}}\right)\right]\sin\omega t + \varepsilon_0^2(x_1 - x_2 - x_3) + \varepsilon_0^4(x_3 - x_2) \tag{219}$$

$$\dot{x}_3 = \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha}\,\widetilde{\text{erf}}\left(\frac{y_3}{\sqrt{2}}\right)\right]\sin\omega t + \varepsilon_0^2(x_1 - x_2 - x_3) + \varepsilon_0^4(x_2 - x_3) \tag{220}$$

If $\varepsilon_0 = 0$, then the solution to this system converges to a three-dimensional stochastic attractor with the joint density given by (63) at $n = 3$.

When $\varepsilon_0 \neq 0$ (but $\varepsilon_0 \to 0$), the solution to this system first converges to a two-dimensional attractor on the plane

$$x_1 = x_2 + x_3 \tag{221}$$

with the joint density given by equation (63) at $n = 3$ after substitution of equation (221) instead of x_1 and redefining the constant from the normalization condition. Then the solution converges to a one-dimensional attractor which dwells on the line

$$x_2 = x_3 \tag{222}$$

of the plane (221).

8. CONCLUSION

This paper presents and discusses physical models for simulating some aspects of neural intelligence, and in particular the process of cognition. The main departure from classical approach here is in utilization of a terminal version of classical dynamics introduced in Zak (1992, 1993*a*). Based upon violations of the Lipschitz condition at equilibrium points, terminal dynamics attains two new fundamental properties: it is spontaneous and nondeterministic. Special attention is focused on terminal neurodynamics as a particular architecture of terminal dynamics which is suitable for modeling information flows. Terminal neurodynamics possesses a well-organized probabilistic structure which can be analytically predicted, prescribed, and controlled, and therefore presents a powerful tool for modeling real-life uncertainties. Two basic phenomena associated with random behavior of neurodynamic solutions are exploited. The first one is a stochastic attractor—a stable stationary stochastic process to which random solutions of a closed system converge. As a model of the cognition process, a stochastic attractor can be viewed as a universal tool for generalization and formation of classes of patterns. The concept of stochastic attractor is applied to model a collective brain paradigm explaining coordination between simple units of intelligence which perform a collective task without direct exchange of information. The second fundamental phenomenon discussed is terminal chaos which occurs in open systems. Applications of terminal chaos to information fusion as well as to explanation and modeling of coordination among neurons in biological systems are discussed. It should be emphasized that all the models of terminal neurodynamics are implementable in analog devices, which means that all the cognition processes discussed are reducible to the laws of Newtonian mechanics.

ACKNOWLEDGMENTS

This research was carried out at the Center for Space Microelectronics Technology, Jet Propulsion Laboratory, California Institute of Technology. Support for the work came from Agencies of the U.S. Department of Defense, including the Innovative Science and Technology Office of the Ballistic Missile Defense Organization, and the US Army/ASAS Project Office, through an agreement with the National Aeronautics and Space Administration. This manuscript was expertly prepared by Annie Aroyan.

REFERENCES

Cetin, B. C., Kerns, D. A., Burdick, J. W., and Barhen, J. (1991). Analog circuits for terminal attractors, repellers and gradient descent, Robotics and Mechanical Systems Report No. RMS-92-01, Department of Mechanical Engineering, California Institute of Technology, Pasadena, California.

Geissler, H. (1987). *Psychological Research*, **49**, 99–116.

Harth, E. (1993). *IEEE Transactions on Systems Man, and Cybernetics*, **13**, 782–789.

Hebb, D. O. (1949). *The Organization of Behavior*, Wiley, New York.

Huberman, B. (1989). *International Journal of Neural Systems*, **1**(1), 41–45.

McCulloch, W. S. (1943). *Bulletin of Mathematical Biophysics*, **5**, 115–133.

Osovets, S., Ginzburg, D., Furfinkel, V. S., Zenkov, L. P., Latash, L. P., Malkin, V. B., Mel'nichuk, P. V., and Pasternak, E. B. (1983). *Uspekhi Fiziki Nauk*, **141**, 103–150.

Rumelhart, D. (1987). *Parallel Distributed Processing*, Cambridge, Massachusetts, Vol. 1, p. 12.

Seeley, T., and Levien, R. (1988). *Sciences*, **1988**(July), 39–42.

Zak, M. (1988). *Physics Letters A*, **133**(1, 2), 18–22.

Zak, M. (1989*a*). *Neural Network* **2**(3), 259–274.

Zak, M. (1989*b*). *Physics Letters A*, **135**(1), 25–28.

Zak, M. (1989*c*). *Complex Systems*, **1989**(3), 471–492.
Zak, M. (1990*a*). *Applied Mathematics Letters*, **3**(3), 131–135.
Zak, M. (1990*b*). *Biological Cybernetics*, **64**(1), 15–23.
Zak, M. (1991*a*). *Biological Cybernetics*, **64**, 343–351.
Zak, M. (1991*b*). *IEEE Expert*, **1991**(August), 4–10.
Zak, M. (1991*c*). *Biological Cybernetics*, **65**, 121–127.
Zak, M. (1992). *International Journal of Theoretical Physics*, **31**(2), 333–342.
Zak, M. (1993*a*). *International Journal of Theoretical Physics*, **32**(1), 159–190.
Zak, M. (1993*b*). Introduction to terminal dynamics, *Complex Systems*, in press.